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An Analysis of Hub Number in Various Fuzzy Graphs

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Abstract

In the present work, we aim to talk about the analytical findings about the quantification of hub structures arising from various graph operations applied to pairwise combinations of connected graphs and paths. Specifically, we delineate the hub numbers resulting from the intersection and joining of two interconnected graphs. We also derive the hub numbers for the intersection of two complete fuzzy graphs, as well as the intersection of a non-exhaustive connected fuzzy graph and a complete fuzzy graph.5 Moreover, we determine the hub configuration for the intersection of two paths, denoted P_n and P_m, whereby $n \geq 2$ and $m \geq 3$. In addition to enumerating these hub values, we provide an upper boundary on the maximum hub number attainable from taking the join of two paths P_n and P_m, where $2 \leq m \leq n$. Through a rigorous mathematical treatment of these graph constructions and evaluations of their associated hub structures, the present work aims to systematically characterize and compare the topological properties induced by different relational combinations of graphs and paths. It is hoped that communication of these findings will offer novel insight into the structural transformations and complexity changes incurred by various graph operations.

Keywords: Hub number, fuzzy graph operations, complete fuzzy graph.

AMS Mathematical Subject Classification [2010]: 05C69, 05C72, 05C75.

1 Introduction

Fuzzy graphs are a generalization of classical graph theory that allows for the representation of uncertainty and vagueness in graphs. They are widely used in various fields such as engineering, economics, computer science, and social disciplines. Fuzzy graphs have applications in decision making, statistics, networking, and modeling real-life issues. They can be used to represent ambiguous networks, analyze network characteristics, and solve real-world problems such as election competitions and finding central affected nodes in infectious diseases [1, 2]. Additionally, fuzzy graphs have been extended to Pythagorean fuzzy sets and Interval-Valued Pythagorean Neutrosophic Graphs (IVPNG) to model human thinking and real-time situations. These extensions introduce new concepts such as regular, strong, product, support strong, and effective

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balanced IVPNGs, which can be used for aggregating information and successful curriculum design [3, 4]. Crisp graphs, being a fundamental mathematical construct, exhibit a plethora of operations that allow for their manipulation and analysis. These operations include but are not limited to, union, intersection, join, tensor product, Cartesian product, composition, strong product, disjunction, and symmetric difference of graphs. A comprehensive treatment of these operations is provided in [5, 6, 7, 8, 9, 10].

The theoretical foundations and notation employed in the present study are informed by the scholarship delineated in the cited references [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Regarding fundamental graph concepts, Mahioub [11] provides a comprehensive study of fuzzy graph theory. Mahioub and Haifa [12, 13] define domiantion in product fuzzy graphs. Mordeson and Nair [14] investigate fuzzy cycles and cocycles. Operations on fuzzy graphs are examined by Mordeson and Peng [15] and Mordeson and Yao [16]. Ore [17] and Rosenfeld [18] establish foundational notions of graphs and fuzzy graphs, respectively. Ramaswmy [19] defines product fuzzy graphs. E. Sampathkumar [20] and Somasundaram and Somasundarm [21] define global domination and domination in fuzzy graphs. Venugopalam and Kumari [22] delineate operations on fuzzy graphs. Zadeh [23, 24] introduces seminal notions of fuzzy sets and similarity relations seminal to this framework. The above sources provide the structural typologies, algebraic formalisms and problem conceptualizations underscoring the analytical objectives and modeling techniques employed herein. Drawing from this extant literature, the key terminology, postulates, and required theoretical apparatus are established.

Furthermore, Ahmed and Shubatah [25] define and compute the hub number of fuzzy graphs. Exploring this concept further, Ahmed and Shubatah [26] introduce the notion of total hub number in fuzzy graphs. Providing antecedent context, Matthew [27] establishes the hub number of traditional graphs. Those sources directly apply hub graph theory to fuzzy graph configurations, thereby furnishing critical context and founding definitions for the present investigation's analytical objectives and modeling approach focused on structural properties induced by graph operations.

Lemma 1.1. *Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining the nodes of V_1 and V_2 where we a assume that $V_1 \cap V_2 \neq \phi$ then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 + G_2 : (\mu_1 + \mu_2, \rho_1 + \rho_2)$ defined as follows:*

$$\mu_1 + \mu_2 = \begin{cases} (\mu_1 \cup \mu_2) & if\ u \in V_1 \cap V_2 \\ \mu_1(u); \quad u \in V_1 - V_2 \\ \mu_2(u); \ u \in V_2 - V_1 \end{cases} \tag{1}$$

and

$$\rho_1 + \rho_2 = \begin{cases} (\rho_1 \cup \rho_2) & if\ (u,\ v) \in E_1 \cap E_2 \\ \rho_1(u,\ v); \quad (uv) \in E_1 - E_2 \\ \rho_2(u,\ v); \quad (u,\ v) \in E_2 - E_1 \\ \mu_1 \times \mu_2 & if (u,\ v) \in E' \end{cases} \tag{2}$$

Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the intersection $G^* = G_1^* \cap G_2^* = (V_1 \cap V_2, E_1 \cap E_2)$ of graphs. We a assume that $V_1 \cap V_2 \neq \phi$ then the intersection of two fuzzy graphs G_1 and G_2 is a Product fuzzy graph $G = G_1 \cap G_2 : (\mu_1 \cap \mu_2\ ,\ \rho_1 \cap \rho_2)$ defined as follows:

$$\mu_1 \cap \mu_2 = \left\{ \ min(\mu_1, \mu_2) \quad if\ u \in V_1 \cap V_2. \right. \tag{3}$$

and

$$\rho_1 \cap \rho_2 = \left\{ \ min(\rho_1, \rho_2) \quad if\ (uv) \in E_1 \cap E_2 \right. \tag{4}$$

Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the union $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ of graphs, where $V_1 \cup V_2 = \phi$ then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \cup G_2 : (\mu_1 \cup \mu_2\ ,\ \rho_1 \cup \rho_2)$ defined as follows:

$$\mu_1 \cup \mu_2 = \begin{cases} max(\mu_1, \mu_2) & if\ u \in V_1 \cup V_2 \\ \mu_1(u);\ u \in V_1 - V_2 \\ \mu_2(u);\ u \in V_2 - V_1 \end{cases} \tag{5}$$

and

$$\rho_1 \cup \rho_2 = \begin{cases} max(\rho_1, \rho_2)(u,\ v) & if\ (u,\ v) \in E_1 \cup E_2 \\ \rho_1(u,\ v);\ (u,\ v) \in E_1 - E_2 \\ \rho_2(u,\ v);\ (u,\ v) \in E_2 - E_1 \end{cases} \tag{6}$$

Theorem 1.2. *[12]*

- *For any complete fuzzy graph $K_\mu, h(K_\mu) = 0$.*
- *For any path fuzzy graphs $p_n, h(p_n) = \sum_{i=1}^{p-2} \mu(v_i)$.*

2 Hub Number in Some Operations on Fuzzy Graphs

In this section, we aim to characterize and examine the structural attribute denoted as the hub number within the context of certain relational operations undertaken on configurations of ambiguous graphs. Specifically, we introduce and analyze the hub number construct for the intersection and joining graph-theoretical constructs applied to combinations of fuzzy graphs.

In the following theorem, we give hub numbers h, γ and γ_g of the intersection of any disjoint fuzzy graphs G_1 and G_2.

Theorem 2.1. *Let G_1 and G_2 be two disjoint fuzzy graphs,*

$$h(G_1 \cap G_2) = 0.$$

Proof. Let H_1 be an $h - set$ of a fuzzy graph G_1 and let H_2 be an $h - set$ of a fuzzy graph G_2. Since G_1 and G_2 are disjoint then $H_1 \cap H_2 = \phi$. Therefore $h(G_1 \cap G_2) = |H_1 \cap H_2| = |\phi| = 0$. □

Theorem 2.2. *Let G_1 and G_2 be two non disjoint fuzzy graphs,*

$$h(G_1 \cap G_2) \leq \gamma(G_1 \cap G_2).$$

Proof. Let $G_1 = V(P_n) = \{v_1, v_2, , v_n\}$, and $G_2 = V(P_m) = \{v_1, v_2, , v_n\}$ be two fuzzy graphs the following three cases are considered: Case 1: When $n = 2$. Suppose that $\{v_1,\ v_2\}$ are the vertices of a path P_2, and $m = 3$ Suppose that $\{v_1,\ u_1,\ u_2\}$ are the vertices of a path P_3 see Figure 3.1, then, $(v_1^* \cap v_2^*) = \{v_1\}$ is not hub set so $h(G_1 \cap G_2) = 0$. and hence, $h(G_1 \cap G_2) < \gamma(G_1 \cap G_2)$.

□

Example 2.3. Consider two fuzzy graphs G_1 and G_2 given in Fig. 1(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ ,\mu_2(u_3) = 0.3$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. $1(b)$ gives the intersection fuzzy graph $(G_1 \cap G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1\}$ defined as: $\mu(v_1) = 0.2$

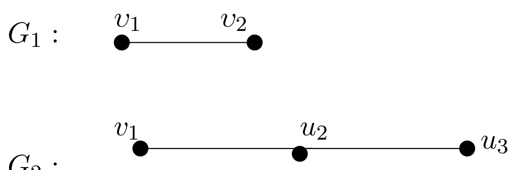


Fig. 1(a): G_1 and G_2

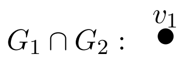


Fig. 1(b): $G_1 \cap G_2$

Case 2: When $n = 2$. Let $\{v_1,\ v_2\}$ be the vertices of a path P_2, and $m = 3$ Suppose that $\{v_1,\ v_2, u_1\}$ are the vertices of a path P_3 see Figure 3.2, then, $(V_1^* \cap V_2^*) = 2 = \{v_1, v_2\}$ is not hub set so, $h(G_1 \cap G_2) = 0$. and hence, $h(G_1 \cap G_2) < \gamma(G_1 \cap G_2)$.

Example 2.4. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 2(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(v_2) = 0.3,\ \mu_2(u_1) = 0.3$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 2(b)) gives the intersection fuzzy graph $(G_1 \cap G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1,\ v_2\}$ defined as: $\mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \rho(v_1,\ \ v_2) = 0.1$.

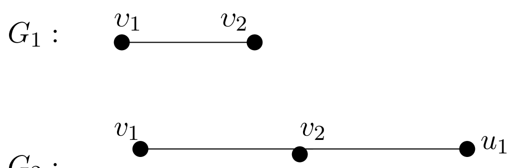


Fig. 2(a): G_1 and G_2

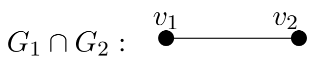


Fig. 2(b): $G_1 \cap G_2$

Case 3: When $n = 3$. Let $\{v_1,\ v_2, v_3\}$ be the vertices of a path P_3, and $m = 4$. Suppose that $\{v_1,\ v_2, v_3, u_1\}$ are the vertices of a path P_4 see Figure 3(b), then, $(V_1^* \cap V_2^* = \{v_1, v_2, v_3\}$, then $D = \{v_2\}$ is a hub set which is also a dominating set.Hence, $h(G_1 \cap G_2) = \gamma(G_1 \cap G_2) = 0.1$ this complete the proof.

Example 2.5. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 3(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(v_2) = 0.3, \mu_2(v_3) = 0.3, \mu_2(u_1) = 0.3$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 3(b)) gives the intersection fuzzy graph $(G_1 \cap G_2) = (V,\ \mu,\ \rho)$ where, $V = \{u_2,\ v_1,\ v_2,\ v_3\}$ defined as: $\mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \mu(v_3) = 0.3,\ \rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$.

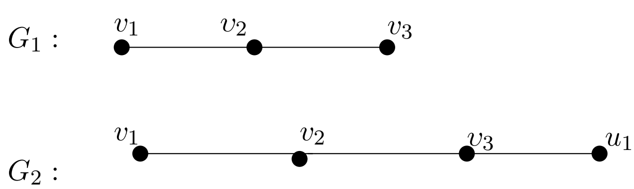


*Fig.*3(a): G_1 and G_2

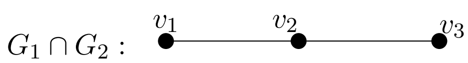


*Fig.*3(b): $G_1 \cap G_2$

Theorem 2.6. *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ be two connected fuzzy graph,*

$$h(G_1+G_2)=\begin{cases} 0\ if\ G_1\ and\ G_2\ are\ complete \\ \mu(v_i),\ if\ G_1\ is\ complete\ and\ G_2\ is\ not\ complete \\ min(h(G_1),h(G_2))\ ifG_1, and\ G_2\ are\ not\ complete \end{cases} \tag{7}$$

Proof. Suppose G_1 and G_2 are both complete. Then G_1+G_2 is also complete fuzzy craph. By (Theorem 2.1), $h(G_1+G_2)=0$. Suppose G_1 is complete and G_2 is non-complete. Let $u\in V(G_1)$ and $H=\{u\}$. Let $v,w\in V(G_1+G_2)-H$. Consider the following cases:
Case 1. Suppose $v,w\in V(G)-u$. Since G_1 is complete fuzzy graph, there is a path, v,u,w in G_1. Hence, there is an H-path between v and w in G_1+G_2.
Case 2. Suppose $v\in V(G_1)-u$ and $w\in V(G_2)$. Since G_1 is complete fuzzy graph, v and u are adjacent. By definition of G_1+G_2, u is adjacent to w. Hence, there is a path, v,u,w in G_1+G_2. Thus, there is an H-path between v and w in G_1+G_2.
Case 3. Suppose $v,w\in V(G_2)$. By definition of G_1+G_2, u is adjacent to both v and w. Hence, there is a path v,u,w in G_1+G_2. Thus, there is an H-path between v and w in G_1+G_2. Thus, H is a hub set of G_1+G_2. Accordingly, $h(G_1+G_2)=\mu(u)$. Since H is non-complete, consequently G_1+G_2 is non-complete. By Theorem[2.1], $h(G_1+G_2)\neq 0$. Therefore, $h(G_1+G_2)=\mu(u)$.
Suppose G_1, and G_2 are both non-complete fuzzy graph. Consider the following cases:
Case 1. Suppose $h(G)=\mu(u)$. Let $u\in V(G_1)$. Let $H=\{u\}$ be a minimum hub set of G_1. Let $v,w\in V(G_1+G_2)-u$. Consider the following subcases:
Subcase 1. Suppose $v,w\in V(G_1)-u$. Since H is a hub set of G_1, there is an H-path between v and w in G_1+G_2.
Subcase 2. Suppose $v\in V(G_1)-u$ and $w\in V(G_2)$. Since H is a hub set of G_1, v is incident to u. By definition of G_1+G_2, u is incident to w. This means that $\{v,u,w\}$ is an $H-path$ in G_1+G_2. Hence, H is a hub set of G_1+G_2. Thus, $h(G_1+G_2)=1$.
Subcase 3. Suppose $v,w\in V(G_2)$. By definition of G_1+G_2, u is incident to both v and w. This means that {v,u,w} is an H-path in G_1+G_2. Hence,H is a hub set of G_1+G_2. Thus, $h(G_1+G_2)=\mu(u)$. Combining the three subcases, $h(G_1+G_2)=\mu(u)$. Since $G_1$and G_2 are both non-complete, G_1+G_2 is non-complete. So, $h(G_1+G_2)\neq 0$. Therefore, $h(G_1+G_2)=\mu(u)$.
Case 2. Suppose $h(G_2)=\mu(u)$. The proof is similar to Case 1.
Case 3. $h(G),h(H)\geq\sum_{i=1}^{p-2}\mu(v_i)$. Let $u\in V(G_1), cV(G_2)$, and $H=\{u,c\}$. Consider the following subcases:
Subcase 1. Let $v,w\in V(G_1)-u$. By definition of G_1+G_2, both v, and w are incident to c. That is, there is an H-path v,c,w in G_1+G_2. Hence, $H=\{u,c\}$ is a hub set of G_1+G_2. So, h$(G_1+G_2)\leq\sum_{i=1}^{p-2}\mu(v_i)$.
Subcase 2. Let $v,w\in V(G_2)-c$. The proof is similar to Subcase 1.
Subcase 3. Let $v\in V(G_1)-u$ and $w\in V(G_2)-c$. By definition of G_1+G_2, v is incident to c, c is incident to u, and u is incident to w. That is, $\{v,c,u,w\}$ is an H-path inG_1+G_2. Hence, $H=\{u,c\}$ is a hub set of G_1+G_2. So, $h(G_1+G_2)\leq\sum_{i=1}^{p-2}\mu(v_i)$. Suppose $h((G_1+G_2)=\mu(u)$. Assume without loss of generality,$H=\{u\}$ be a minimum hub set of G_1+G_2, where $u\in V(G_1)$. Let $v,w\in V(G_1)-u$. Thus, $\{v,u,w\}$ is an H-path in G_1. This implies that H is a hub set of G_1. That is, $h(G_1)=\mu(u)$. This is a contradiction to the assumption that $h(G_1)\geq\sum_{i=1}^{p-2}\mu(v_i)$. Therefore, $h(G_1+G_2)=\sum_{i=1}^{p-2}\mu(v_i)$. □

Corollary 2.7. *Let G_1 and G_2 are non complete fuzzy graphs at least one of the following holds*
(ii)

$$\gamma(G_1+G_2)\leq h(G_1+G_2).$$

(ii)

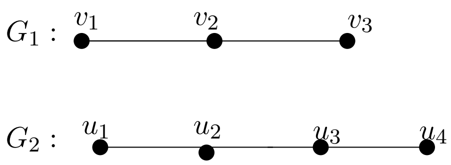
$$\gamma(\overline{G_1 + G_2}) \leq h(G_1 + G_2).$$

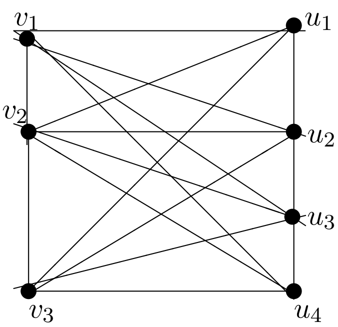
Proof. Let G_1 and G_2 are non complete fuzzy graphs and let H be an $H - set$ of $G_1 + G_2$ then H is a dominating set of $G_1 + G_2$. Hence,

$$\gamma(G_1 + G_2) \leq h(G_1 + G_2).$$

similarly, (ii) holds. □

Example 2.8. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 4(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(u_1) = 0.4,\ \mu_2(u_2) = 0.3, \mu_2(u_3) = 0.5, \mu_2(u_4) = 0.4$, and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. $4(b)$) gives the join fuzzy graph $(G_1 + G_2) = (V,\ \mu,\ \rho)$ where, $V = \{u_1,\ u_2,\ u_3 u_4,\ v_1,\ v_2, v_3,\}$ defined as: $\mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \mu(v_3) = 0.3,\ \mu_2(u_1) = 0.4,\ \mu_2(u_2) = 0.3, \mu_2(u_3) = 0.5, \mu_2(u_4) = 0.4 \rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_2$.





(100,120)

Fig. $4(a)$: G_1 and G_2 *Fig.* $4(b)$: $G_1 + G_2$

We see that $h(G_1 + G_2) = \gamma(G_1 + G_2) = 0.1$

Theorem 2.9. *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ are non complete fuzzy graphs , $h(\overline{G_1 + G_2}) \neq h(\overline{G_1} + \overline{G_2})$.*

Example 2.10. Consider the fuzzy graphs $G_1 = (V_1,\ \mu_1,\ \rho_1)$ where, $V_1 = \{v_1,\ v_2,\ v_3\}$, $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$, $G_2 = (V_2,\ \mu_2,\ \rho_2)$ where, $V_2 = \{u_1, u_2\}$, $\mu_2(u_1) = 0.4,\ \mu_2(u_2) = 0.3,\ \rho(u_1, u_2) = 0.12$, G $= (G_1 + G_2) = (V,\ \mu,\ \rho)$ where, $V = V_1 \cup V_2 = \{u_1, u_1, v_1, v_2, v_3\}$, $\mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \mu(v_3) = 0.3,\ \mu(u_1) = 0.4,\ \mu(u_2) = 0.3,\ \rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V$ $\overline{G_1 + G_2} = (V, \overline{\mu}, \overline{\rho})$ where, $V = \{u_1, u_1, v_1, v_2,\ v_3\}$, $\overline{\mu}(v_1) = 0.2,\ \overline{\mu}(v_2) = 0.1,\ \overline{\mu}(v_3) = 0.3,\ \ \overline{\mu}(u_1) = 0.4,\ \overline{\mu}(u_2) = 0.3,\ \overline{\rho}((u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V$ *and* $\overline{\rho}(v_1,\ v_2) = \overline{\rho}(u_1,\ v_3) = \ \overline{\rho}(u_2,\ v_2) = \ \overline{\rho}(u_2,\ v_3) = \ \overline{\rho}(u_2,\ v_1) = \ \overline{\rho}(u_1,\ v_3) = \ \overline{\rho}(u_1,\ v_1) = \ \overline{\rho}(u_1,\ v_2) = \ \overline{\rho}(u_1,\ u_2) = 0$, $\overline{G_1} = (V, \overline{\mu}_1, \overline{\rho}_1)$ where, $V = \{v_1,\ v_2,\ v_3\}$, $\overline{\mu}_1(v_1) = 0.2,\ \overline{\mu}_1(v_2) = 0.1,\ \overline{\mu}_1(v_3) = 0.3,\ \ \rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V$ $\overline{G_2} = (V,\ \overline{\mu}_2,\ \overline{\rho}_2)$ where, $V_2 = \{u_1, u_2\}$, $\overline{\mu}_1(u_1) = 0.4,\ \overline{\mu}_1(u_2) = 0.3$ and $\overline{\rho}_2(u_1,\ u_2) = 0$, which given in Fig. $5(a),\ 5(b),\ 5(c),\ 5(d)$, respectively. Finally $G = \overline{G_1} + \overline{G_2} = (V, \overline{\mu}, \overline{\rho})$ given in Fig. 5(e).

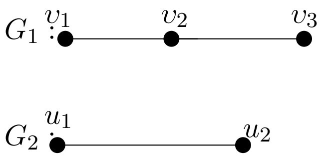


Fig. 5(*a*): G_1 and G_2

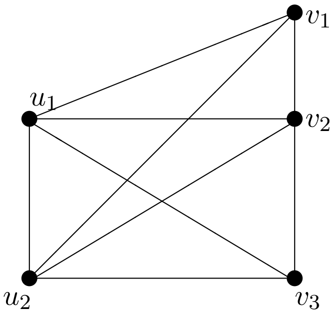


Fig. 5(b): $G_1{+}G_2$

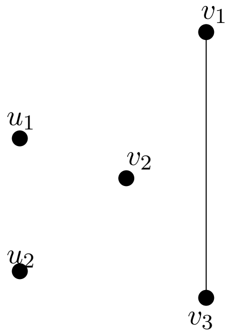


Fig. 5(c): $\overline{G_1 + G_2}$

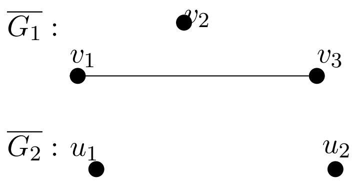


Fig. 5(*d*): $\overline{G_1}$ and $\overline{G_2}$

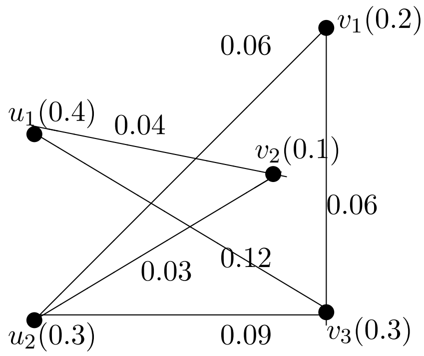


Fig. 5(*e*): $\overline{G_1}{+}\overline{G_2}$

We see that, $h(\overline{G_1 + G_2}) \neq h(\overline{G_1} + \overline{G_2})$.

Theorem 2.11. *If $G_1 = p_n = (\mu_1,\ \rho_1)$ and $G_2 = p_m = (\mu_2,\ \rho_2)$ be two path fuzzy graphs such that $\rho_1(u,v) = \mu_1(u) \wedge \mu_2(v)$ for all $(u,v) \in E_1$, $\rho_2(u,v) = \mu_2(u) \wedge \mu_2(v)$, for all $(u,v) \in E_2$, then*

$$h(G_1 \cup G_2) = \begin{cases} \sum_{i=1}^{p-2} \mu(v_i)\ if V_1^* \cap V_2^* = 1 \\ \sum_{i=1}^{p-3} \mu(v_i)\ if V_1^* \cap V_2^* > 1\ and\ n, m \geq 3 \end{cases} \tag{8}$$

Proof. Let $G_1 \cup G_2$ be the union of the fuzzy graphs G_1 and G_2. Then, $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

Case 1: If $V_1^* \cap V_2^* = 1$ This means the fuzzy connected components of G_1 and G_2 intersect at only one vertex. Then, the connectivity of $G_1 \cup G_2$ is contributed by all vertices except the vertex in the intersection. Hence, $h(G_1 \cup G_2) = \sum_{i=1}^{p-2} \mu(v_i)$.

Case 2: If $V_1^* \cap V_2^* > 1$ This means the fuzzy connected components intersect at more than one vertex. Then, the connectivity of $G_1 \cup G_2$ is contributed by all vertices except the vertices in the intersection. Hence, $h(G_1 \cup G_2) = \sum_{i=1}^{p-3} \mu(v_i)$.

Therefore, the given formula holds for the height of the union fuzzy graph $G_1 \cup G_2$ based on the intersection of the fuzzy connected components of G_1 and G_2. □

Example 2.12. Consider the two path fuzzy graphs $G_1 = p_n$ and $G_2 = p_m$ given in *Fig.* 6(*a*) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.6,\ \mu_1(v_2) = 0.1$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.5,\ \mu_2(u_1) = 0.6, \mu(u_2) = 0.7$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 6(*b*)) gives the union fuzzy graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1, v_2, u_1, u_2\}$ defined as: $\mu(v_1) = 0.6,\ \mu(v_2) = 0.1$, $\mu(u_1) = 0.6, \mu(u_2) = 0.7$ and $\rho(u,\ v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$

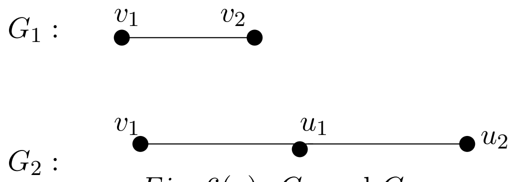


Fig. 6(*a*): G_1 and G_2

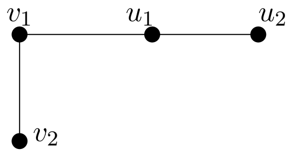


Fig. 6(*b*): $G_1 \cup G_2$

Example 2.13. Consider the two path fuzzy graphs G_1 and G_2 given in *Fig.* 7(*a*) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.5,\ \mu_2(u_1) = 0.1, \mu(u_2) = 0.9$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 7(b)) gives the union fuzzy graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1, v_2, v_3, u_1, u_2\}$ defined as: $\mu(v_1) = 0.5,\ \mu(v_2) = 0.1, \mu(v_3) = 0.3$, $\mu(u_1) = 0.1, \mu(u_2 = 0.9$ and $\rho(u,\ v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$

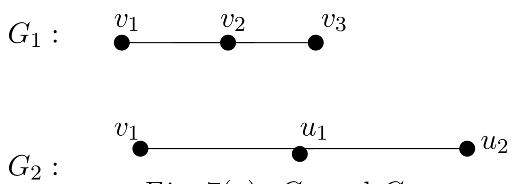


Fig. 7(*a*): G_1 and G_2

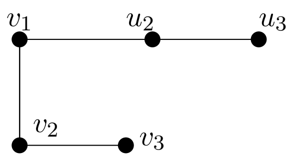


Fig. 7(*b*): $G_1 \cup G_2$

Example 2.14. Consider the two path fuzzy graphs G_1 and G_2 given in Fig. 8(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3, \mu_1(v_4) = 0.5$, and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(u_1) = 0.3, \mu(u_2) = 0.4$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 8(b)) gives the union fuzzy graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1, v_2, v_3, v_4, u_1, u_2\}$ defined as: $\mu(v_1) = 0.4,\ \mu(v_2) = 0.3, \mu(v_3) = 0.3, \mu(v_4) = 0.5,\ , \mu(u_1) = 0.3, \mu(u_2 = 0.4$ and $\rho(u,\ v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$

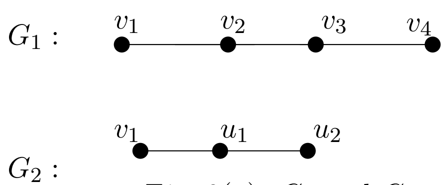


Fig. 8(*a*): G_1 and G_2

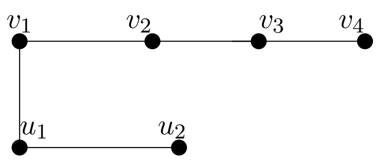


Fig. 8(*b*): $G_1 \cup G_2$

case2: if $V_1 \cap V_2 > 1$

Example 2.15. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 9(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(v_2) = 0.3,\ \mu_2(u_1) = 0.3$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 9(b)) gives the union fuzzy graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1,\ v_2, v_3, u_1\}$ defined as: $\mu(v_1) = 0.4,\ \mu(v_2) = 0.3, \mu(v_3) = 0.3,\ \mu(u_1) = 0.3,\ \rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V$

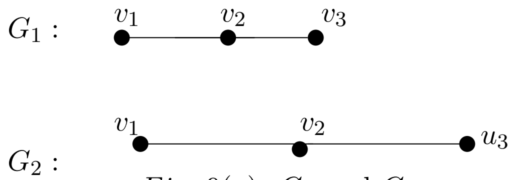


Fig. 9(*a*): G_1 and G_2

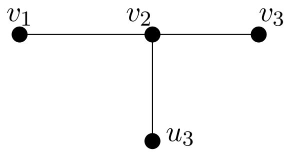


Fig. 9(*b*): $G_1 \cup G_2$

Example 2.16. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 10(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3, \mu_1(v_4) = 0.5,\ \mu_1(v_5) = 0.6$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(v_2) = 0.3, \mu(v_3) = 0.4, \mu(v_4) = 0.5,\ \mu_2(u_1) = 0.3, \mu_2(u_2) = 0.7$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 10(b)) gives the fuzzy union graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1, v_2, v_3, v_4, u_1\}$ defined as: $\mu(v_1) = 0.4,\ \mu(v_2) = 0.3, \mu(v_3) = 0.4, \mu(v_4) = 0.5,\ , \mu(u_1) = 0.3$, and $\rho(u,\ v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$

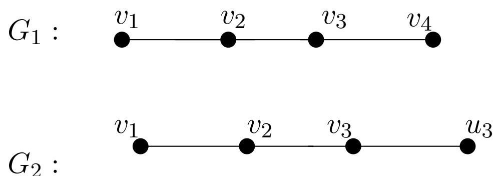


Fig. 10(*a*): G_1 and G_2

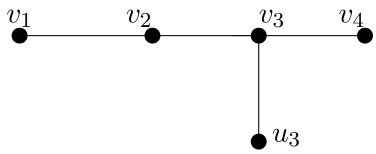


Fig. 10(*b*): $G_1 \cup G_2$

Example 2.17. Consider the two fuzzy graphs G_1 and G_2 given in Fig. 11(a) where, $G_1 = (\mu_1,\ \rho_1)$ defined as: $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3, \mu_1(v_4) = 0.5,\ \mu_1(v_5) = 0.6$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V_1$ and $G_2 = (\mu_2,\ \rho_2)$ defined as: $\mu_2(v_1) = 0.4,\ \mu_2(v_2) = 0.3,\ \mu_2(v_3) = 0.8\ \mu_2(u_1) = 0.3, \mu_2(u_2) = 0.7$ and $\rho(u,\ v) = \mu_2(u) \wedge \mu_2(v)$ for all $u, v \in V_2$. Then the Fig. 11(b)) gives the union fuzzy graph $(G_1 \cup G_2) = (V,\ \mu,\ \rho)$ where, $V = \{v_1,\ , v_3, v_4, v_5, u_1, u_2\}$ defined as: $\mu(v_1) = 0.2,\ \mu(v_2) = 0.4, \mu(v_3) = 0.8, \mu(v_4) = 0.5,\ \mu(v_5) = 0.6, \mu(u_1) = 0.3, \mu(u_2) = 0.7$ and $\rho(u,\ v) = \mu_1(u) \wedge \mu_1(v)$ for all $u, v \in V$

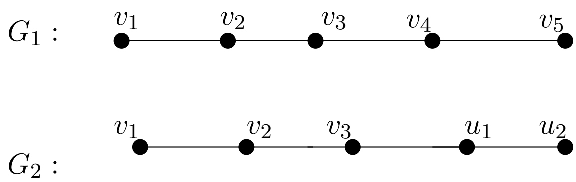


Fig. 11(*a*): G_1 and G_2

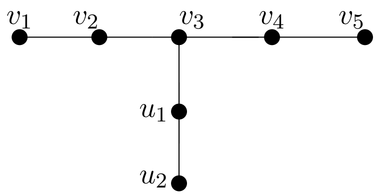


Fig. 11(*b*): $G_1 \cup G_2$

Conclusion

This study aimed to systematically analyze hub structures arising from various graph operations applied to interconnected graphs and paths. Analytical findings were delineated regarding the quantification of hub numbers resulting from intersections and joinings of two connected graphs. Specifically, the hub configurations induced by intersecting two complete fuzzy graphs and intersecting a non-exhaustive connected fuzzy graph with a complete fuzzy graph were mathematically derived. Additionally, the hub topology for intersecting two paths, P_n and P_m, where $n \geq 2$ and $m \geq 3$, was determined. An upper boundary on the maximum attainable hub number from taking the join of paths P_n and P_m, where $2 \leq m \leq n$, was also established. Through a rigorous treatment of the mathematics underlying these graph constructions, key transformations, and complexity changes to topological properties incurred by different relational combinations were characterized and compared. Communication of these analytical findings regarding hub quantifications offers novel theoretical insights into structural modifications induced by graph operations. Overall, the study provided a systematic framework for understanding hub structures emerging from pairwise graph intersections and joins.

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