8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

Stress indices of graphs

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Abstract

Let $G = (V, E)$ be a simple and connected graph. stress of a vertex in a graph G is denoted by $st(v)$ is defined as the number of shortest paths passing through internal vertex v. In this paper we have obtain stress-sum index $\mathcal{SS}(G)$ and second stress index $\mathcal{S}_2(G)$ for standard graphs.

Keywords: Stress of a vertex, stress of a graph, k- stress regular graph.

AMS Mathematical Subject Classification [2010]: 05C12, 05C05

1 Introduction

Centrality measure plays a very important role in the study of network analysis[8]. In 1953, Shimbel first defined the measure of stress centrality based on the shortest path. Stress centrality measure has lot of applications in social, biological networks. Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant.

In this paper we consider simple, connected graph of order n and size m. The concept of stress of a vertex in a graph was introduced by A. Shimbel[2]. The stress is a vertex centrality measure denoted as $st(v)$ and defined as the number of shortest paths in the graph G passing through the internal vertex v and the stress of a graph G is denoted by $st(G)$ and defined as $st(G) = \sum_{v \in V(G)} st(v)$. The line joining the vertices u and v is denoted as uv. The shortest uv path is called geodesic of a graph G. (S. Arumugam) A graph is said to be k-stress regular if all of its vertices have stress k. For standard terminology and notation in graph theory, we follow the text-books [1, 7]. Bhargava et al., Raksha Poojary et al.[3, 5] and are two publications in which the computation of stress of a vertex have been studied in different graphs. The Somber index was defined by Gutman in [10] as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$ where $d_G(u)$ represents the degree of vertex u. Rajendra et al. have introduced stress-sum index $\mathcal{SS}(G)$ and second stress index $\mathcal{S}_2(G)$ for graphs in [12] defined as,

$$\mathcal{SS}(G) = \sum_{uv \in E(G)} [st(u) + st(v)] \tag{1}$$

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and

$$\mathcal{S}_2(G) = \sum_{uv\in E(G)} [st(u)st(v)]\,. \tag{2}$$

Eqn. (1) is equivalently given as,

$$\mathcal{SS}(G) \;=\; \sum_{v\in V(G)} [d(v)st(v)]$$

2 Computation of stress connectivity indices

The stress of each vertex in Peterson graph is 3. Hence $\mathcal{S}_2(P) = 150$. By Eqs.(1) and (2) we have following observation.

Observation 1. For any connected graph G, then the following statements holds good.

1. $st(G) \leq \mathcal{SS}(G)$.
   Equality holds if and only if graph G is complete graph K_n.
2. In general, $\mathcal{SS}(G)$ and $\mathcal{S}_2(G)$ both are independent.
3. In general, st(G) and $\mathcal{S}_2(G)$ both are independent.
4. If $\mathcal{SS}(G) = 0$. if and only if G is complete graph K_n.
5. If $\mathcal{S}_2(G) = 0$ then graph G not necessarily complete graph K_n.

Theorem 2.1. *Let G be a connected regular graph of degree r.*
Then

$$\mathcal{SS}(G) = rst(G).$$

Proof. We have

$$\begin{aligned}\mathcal{SS}(G) &= \sum_{uv\in E(G)} [st(u) + st(v)]\\ &= \sum_{v\in V(G)} [d(v)st(v)]\\ &= r\sum_{v\in V(G)} [st(v)]\end{aligned}$$

therefore,

$$\mathcal{SS}(G) = rst(G).$$

Hence the proof.

□

Corollory 1. For a connected regular graph of degree r having n points with diameter 2 then $\mathcal{SS}(G) = nr^2$.

Corollory 2. For a cycle graph C_n on n vertices, $\mathcal{SS}(C_n)= nd(d-1)$ where $d= \lfloor \frac{n}{2} \rfloor$.

Corollory 3. Peterson graph P is a regular graph of degree 3. Hence $\mathcal{SS}(P)= 90$.

Corollory 4. For a complete graph K_n, n≥2 then $\mathcal{SS}(K_n)= \mathcal{S}_2(K_n)$=0.

Proposition 2.2. *For a cycle graph C_n on n vertices, $\mathcal{S}_2(C_n)= \frac{n}{4}d^2(d-1)^2$ where $d= \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be a vertex set of C_n. For any vertex v_i, we have $st(v_i)= \frac{d(d-1)}{2}$ where $d = \lfloor \frac{n}{2} \rfloor$
then

$$\begin{aligned}\mathcal{S}_2(C_n) &= \sum_{uv \in E(C_n)} [st(u)st(v)] \\ &= \sum_{v_i \in V(C_n)} [st(v_i)]^2\end{aligned}$$

therefore,

$$\mathcal{S}_2(C_n) = \frac{nd^2(d-1)^2}{4}.$$

Hence the proof. □

Proposition 2.3. *For a path P_n on n vertices,*

$$\mathcal{SS}(P_n) = 2\left({}^nC_3\right)$$

and

$$\mathcal{S}_2(P_n) = \frac{n(n-1)}{2}\left[\frac{(n^2+n-1)(2n-1)}{3} - n^2(n-1) + \frac{(2n-1)(3n^2-3n-1)}{15} + (n-n^2)\right].$$

Proof. Let u_1, u_2, ..., u_n be the vertices of P_n. Since u_i is adjacent to u_{i+1}, $i = 1, 2, 3, ..., n-1$.
We have $st(P_n)= {}^nC_3$ and $st(u_i) = (i-1)(n-i)$
Therefore

$$\begin{aligned}\mathcal{SS}(P_n) &= \sum_{i=1}^{n-1} [st(u_i) + st(u_{i+1})] \\ &= 2\sum_{i=1}^{n-1} [st(u_i)] \\ &= 2\left({}^nC_3\right).\end{aligned}$$

$$\begin{aligned}\mathcal{S}_2(P_n) &= \sum_{i=1}^{n-1}[st(u_i)st(u_{i+1})]\\ &= \sum_{i=1}^{n-1}\big[(i-1)(n-i)i(n-i-1)\big]\\ &= \sum_{i=1}^{n-1}\big[(n^2+n-1)i^2-2ni^3+i^4+(n-n^2)i\big]\\ &= (n^2+n-1)\sum_{i=1}^{n-1}i^2-2n\sum_{i-1}^{n-1}i^3+\sum_{i=1}^{n-1}i^4+(n-n^2)\sum_{i=1}^{n-1}i\\ &= \frac{n(n-1)}{2}\left[\frac{(n^2+n-1)(2n-1)}{3}-n^2(n-1)+\frac{(2n-1)(3n^2-3n-1)}{15}+(n-n^2)\right].\end{aligned}$$

Hence the proof. □

Proposition 2.4. *For a complete bipartite graph $K_{m,n}$,*

$$\mathcal{SS}(K_{m,n})=\frac{mn}{2}\big[n(n-1)+m(m-1)\big]$$

and

$$\mathcal{S}_2(K_{m,n})=\frac{1}{2}\big[mn^2(m-1)(n-1)\big].$$

Proof. In a complete bipartite graph $K_{m,n}$, the vertex set $V(K_{m,n})$ can be partitioned into two distinct sets namely $A=\{u_1,u_2,...,u_m\}$ and $B=\{v_1,v_2,...,v_n\}$. Stress of any vertex in a complete bipartite graph $K_{m,n}$ is given by,

$$st(v)=\begin{cases}\frac{n(n-1)}{2} & : \text{ if } v\in\text{A}\\ \frac{m(m-1)}{2} & : \text{ if } v\in\text{B}\end{cases}$$

For $i=1,2,...,m$ and $j=1,2,3,...,n$, every edge u_iv_j in $E(K_{m,n})$, in which $u\in$A and $v\in$B.
Consider

$$\begin{aligned}\mathcal{SS}(K_{m,n}) &= \sum_{u_iv_j\in E(K_{m,n})}[st(u_i)+st(v_j)]\\ &= \sum_{i=1}^{m}st(u_i)+\sum_{j=1}^{n}st(v_j)\\ &= \frac{mn}{2}\big[n(n-1)+m(m-1)\big].\end{aligned}$$

And

$$\begin{aligned}\mathcal{S}_2(K_{m,n}) &= \sum_{u_iv_j\in E(K_{m,n})}[st(u_i)st(v_j)]\\ &= \frac{1}{2}\big[mn^2(m-1)(n-1)\big].\end{aligned}$$

Hence the proof. □

Proposition 2.5. *For a wheel graph $W_{n+1}=C_n+K_1$, $n\geq 4$ on $n+1$ vertices,*

$$\mathcal{SS}(W_{n+1})=3({}^{n}C_2)+\frac{n(n-3)^2}{2}$$

and

$$\mathcal{S}_2(W_{n+1}) = n + \frac{n^2(n-3)}{2}.$$

Proof. A wheel graph W_{n+1}= $C_n + K_1$ where $V(K_1)$= $\{x\}$.
Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the vertex set of cycle C_n.
We have
$st(v_i) = 1$ ∀i, st(x)= $\frac{n(n-3)}{2}$ and $st(W_{n+1})$=$^nC_2$.
Therefore

$$\begin{aligned}\mathcal{SS}(W_{n+1}) &= \sum_{uv \in E(W_{n+1})} [st(u) + st(v)] \\ &= 3st(W_{n+1}) + (n-3)st(x) \\ &= 3(^nC_2) + \frac{n(n-3)^2}{2}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2(W_{n+1}) &= \sum_{uv \in E(W_{n+1})} [st(u)st(v)] \\ &= n + \frac{n^2(n-3)}{2}.\end{aligned}$$

Hence the proof. □

Corollory 5. Let W_{n+1}= $C_n + K_1$. Then the following statement holds good.

1. $\mathcal{SS}(W_{n+1}) \geq st(C_n)$, for n≥3.

2. $\mathcal{S}_2(W_{n+1}) \geq st(C_n)$, for n≥3.

Proposition 2.6. *For a fan graph* F_{n+1}= $P_n + K_1$*,* $n \geq 3$ *on* $n+1$ *vertices,*

$$\mathcal{SS}(F_{n+1}) = \frac{(n-2)\big[n(n-1)+6\big]}{2}$$

and

$$\mathcal{S}_2(F_{n+1}) = n - 3 + \frac{n(n-1)(n-2)}{2}.$$

Proof. A fan graph F_{n+1}= $P_n + K_1$ where $V(K_1)$= $\{x\}$.
Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ be the vertex set of path P_n.
We have

$$st(v_i) = \begin{cases} 0 & : \text{if } i = 1, n \\ 1 & : \text{if } 2 \leq i \leq n-1 \end{cases}$$

and

$$st(x) = \frac{(n-1)(n-2)}{2}.$$

Therefore

$$\begin{aligned}\mathcal{SS}(F_{n+1}) &= \sum_{uv\in E(F_{n+1})}[st(u)+st(v)] \\ &= \frac{n-2}{2}\big[n(n-1)+6\big]\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2(F_{n+1}) &= \sum_{uv\in E(F_{n+1})}[st(u)st(v)] \\ &= n-3+\frac{n(n-1)(n-2)}{2}.\end{aligned}$$

□

Corollary 6. Let $F_{n+1}= C_n + K_1$. Then the following statement holds good.

1. $\mathcal{SS}(F_{n+1}) \geq st(C_n)$, for n≥3.
2. $\mathcal{S}_2(F_{n+1}) \geq st(C_n)$, for n≥3.

Proposition 2.7. *For a friendship graph(or windmill graph) F_n, $n \geq 2$ on $2n+1$ vertices,*

$$\mathcal{SS}(F_n) = 4n^2(n-1)$$

and

$$\mathcal{S}_2(F_n) = 0.$$

Proof. In a friendship graph central vertex has stress $2n(n-1)$ and remaining $2n$ vertices have stress 0. Therefore

$$\mathcal{SS}(F_n) = 4n^2(n-1)$$

and

$$\mathcal{S}_2(F_n) = 0.$$

□

Proposition 2.8. *For a star graph $K_{1,n}$ on $n+1$ vertices,*

$$\mathcal{SS}(K_{1,n}) = \frac{n^2(n-1)}{2}.$$

and

$$\mathcal{S}_2(K_{1,n}) = 0.$$

Proof. In a star graph $K_{1,n}$, internal vertex has stress $\frac{n(n-1)}{2}$ and remaining n vertices have stress 0. Therefore

$$\mathcal{SS}(K_{1,n}) = \frac{n^2(n-1)}{2}$$

and

$$\mathcal{S}_2(K_{1,n}) = 0.$$

Hence the proof. □

Proposition 2.9. *For a bistar graph $A(n,k)$ on n vertices,*

$$\mathcal{SS}\left(A(n,k)\right) = \frac{k(k+1)(2n-k-3) + \left((n-1)^2 - k^2\right)(n-k-2)}{4}$$

and

$$\mathcal{S}_2\left(A(n,k)\right) = \frac{k(2n-k-3)(n-k-2)(n+k-1)}{4}.$$

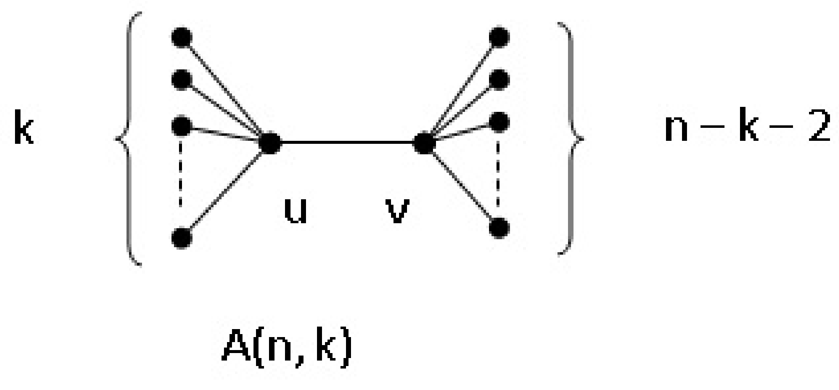

Figure 1

Proof. Let $A(n,k)$ be the tree as shown in the Fig. 1. Then $st(u) = \frac{k}{2}(2n-k-3)$ and $st(v) = \frac{1}{2}(n-k-2)(n+k-1)$ by computation the result follows. $\square$

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