On the Geodetic Number of a Logistics Network

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ABSTRACT

The work is devoted to the geodetic number of a graph in logistics terms. A logistics model is introduced based on the concept of the geodetic number of a graph. The geodetic numbers are found for the graphs of the five regular polyhedra and the lattice graphs. An application of a bipyramidal structure in tourism logistics is demonstrated.

KEYWORDS: logistics network, graph, polyhedron, lattice, geodetic number

1 INTRODUCTION

Innovative service technologies in tourism include, but not limited to, the followings: organizing catering, excursions, expanding the number of tourist attractions, etc. For the improvement of service quality, it is necessary to elaborate optimal transportation schemes between a given number of nodes connected by some network (graph), in which the problem of finding optimal logistics models comes first. When constructing optimal transportation plans, there arise problems of optimizing time-dependent or multi-path processes, and thus it is important to study and construct special communication schemes for which optimization problems are effectively solved.

This study proposes a new logistics model based on the concept of geodetic generators. With regard to logistics, the main advantage of this model is that the manager can concentrate on running logistics only between a number of main nodes of the network (main transportation hubs), called geodetic generators. From the optimization standpoint, it is convenient to deliver resources to the rest of the nodes of the network because each node of the network is located on some geodesic path connecting two of the generators. Methodologically, we follow the approach outlined in [1].

2 THE PROBLEM STATEMENT

We solve the problem of finding geodetic generators of the graph of each regular polyhedra as well as other important graphs. Historically, the concept of geodetic number of a connected graph $G=(V,E)$ and the concept of geodetic generators of G were first introduced by Harary, Loukakis and Tsouros [2]. Let $S\subset V$ be some set of vertices of G. The geodesic closure of S is defined to be the set of all vertices $u\in V$ that appear on geodesic paths in G, connecting all possible pairs of nodes in S. If the closure of S contains all the vertices of G, then the nodes in S are called *geodetic generators*. The *geodetic number* of G is denoted $g(G)$ and is defined to be the minimum size of S, that is, the minimum number of generators. In [7] it is shown that finding $g(G)$ is an NP-hard problem and an algorithm is given for finding $g(G)$. By now, the geodesic numbers are known for some classes of graphs; see [2, 3, 4] and references therein. In particular, in [4], estimates are obtained for a variation of the geodetic number of G called the *strong edge geodesic number*, and an application of this number is presented for constructing a patrol scheme for an urban road network.

The geodetic number of a graph introduced in [2] has two well-known [5] applications; one in location theory and the other in convexity theory. In this paper, the geodetic number is also used in tourism logistics and is identified for the graphs of the five regular polyhedra and for the lattice graphs.

3 REGULAR POLYHEDRA

In this section, we show that for the graphs of all regular centrally symmetric polyhedra, i.e., the graphs of the octahedron, cube, dodecahedron and icosahedron (the tetrahedron is an exception since it has no center of symmetry), the geodetic number is 2; moreover, as two generators there can be taken any pair of centrally symmetric vertices. Note first that the geodetic number of the tetrahedron is equal to 4 since the graph of this polyhedron is a complete graph on 4 vertices [2].

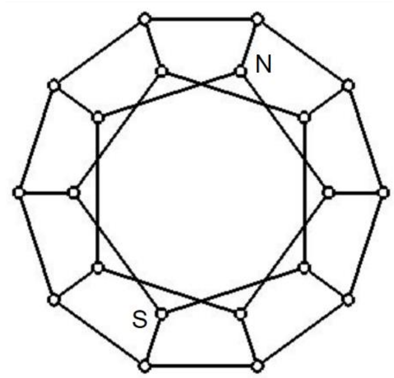

Figure 1: Dodecahedron with a pair of centrally symmetrical vertices, N and S

Consider the other four regular polyhedra, starting with the dodecahedron. Vertices N and S of the dodecahedron, chosen to be symmetrical with respect to the center of the dodecahedron, are conventionally associated with the north and south poles, respectively. One of possible choices of N and S is shown in Figure 1. The graph-theoretic distance between vertices N and S is equal to 5 in the sense that these vertices can be connected by a *path of length* 5, i.e., a (simple) path of 5 edges of the graph, but N and S cannot be connected by a path with fewer edges. It is easy to check that each vertex of the dodecahedron is included in some path of length 5, connecting N and S. Thus, vertices N and S are geodetic generators of the graph of the dodecahedron, and the geodetic number is equal to 2. Similarly, we have shown that the geodetic number of the cube, octahedron, and icosahedron is also equal to 2, and in each case we can take any pair of centrally symmetric vertices of the corresponding polyhedron as geodetic generators. Clearly, the geodetic number of the graph of a polyhedron is equal to 2 whenever the polyhedron has the following two properties:

1) the polyhedron has a center of symmetry, and
2) one can cover all vertices of the polyhedron by the action of the stabilizer of the set $\{N, S\}$ on one geodesic path connecting N and S.

Note that the geodetic number is equal to 2 not only for the graph of the octahedron, but also for the graph of any so-called *bipyramid*, i.e., a polyhedron with a bipyramidal structure, in which two vertices can be distinguished (N and S) so that any other vertex of the polyhedron is adjacent either to N or to S (or both); see [6]. An ordinary octahedron is an example of one of the simplest bipyramids. Bipyramids of general form are also studied in [6]; in particular, a toroidal bipyramid with 8 vertices is mentioned, which has two polyhedral realizations having the same graph but not having a single common face [7].

Bipyramidal graphs can be used in the design of tourist attractions; for example, the network in Fig. 2 with three routes between the hotel (N) and airport (S), each passing through a respective tourist attraction—A1, A2, A3—so that a tourist is able to choose an attraction according to his needs and budget for an appropriate additional fee. Let us recall that there are different types of tourist attractions; for example, cultural-historical, recreational, service (for example, visiting a restaurant with exotic cuisine), anthropological (ethnic), event-based, and mythological [8]. Thus, each tourist is able to choose the attraction that suits him, while the tourism business will be able to generate surplus.

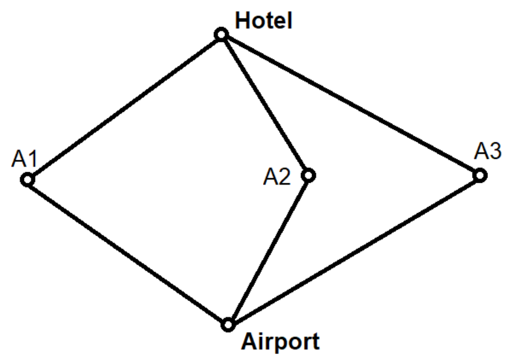

Figure 2: The simplest bipyramidal scheme in which A1, A2, and A3 are tourist attractions of three different types

4 LATTICE GRAPHS

In this section, we determine the geodetic numbers of the graphs of cylindrical and toroidal lattices. The case of a plane rectangular lattice is addressed in [2]; its size is 4×4 (that is, four rows and four columns of quadrilateral regions) and its geodetic number is equal to 2; such a lattice can serve as a model for the road network in a city such as Manhattan.

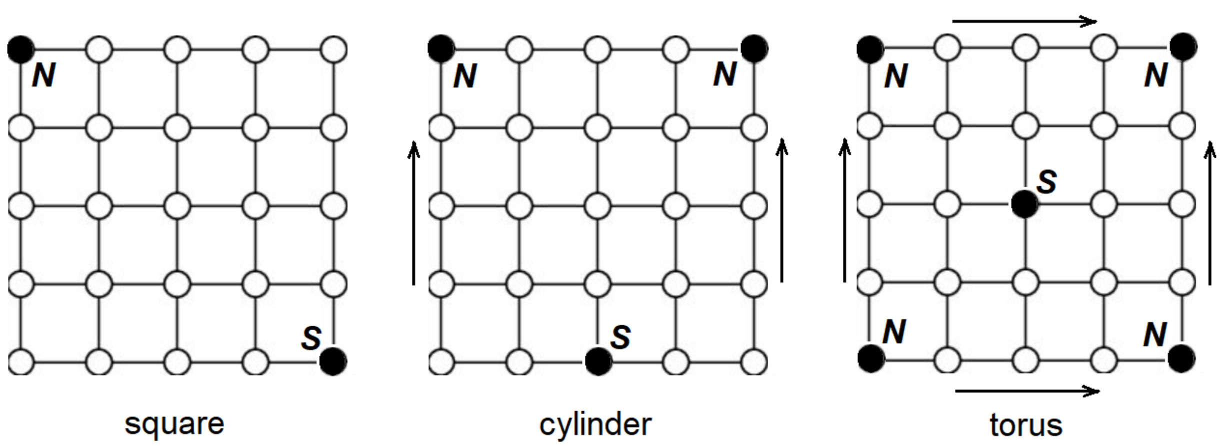

Figure 3: The lattice graphs on the topological disk in the shape of a square (left), cylinder (middle), and the torus (right)

It is mentioned in [2] that the geodetic number of any plane lattice graph is equal to 2, in which case a possible pair of generators is denoted by N and S in Figure 3 (left). Pairs of generators N and S are also shown (respectively) for the cylindrical lattice in Figure 3 (middle) and the toroidal one in Figure 3 (right); to obtain actual cylinder, or the torus, one should identify, in pairs, the opposite sides of the rectangles as indicated by arrows in Figure 3, middle and right, respectively. In general, it can be shown that in the case in which n is even, the graph of any $n \times n$ lattice in the plane, cylinder, or the torus has geodetic number equal to 2. Besides, we can show that if n is odd, then the geodetic number of the graph of any $n \times n$ lattice is equal to 3 or 5 in the cylinder or torus, respectively.

It is interesting to turn this combinatorial problem into a geometric one. For that, we shall measure the Manhatten distance between two nodes of a network not just as the number of edges in a geodesic path connecting those nodes but as the sum of the geometric lengths of those edges in an appropriate geometric realization. The easiest way is to realize a given plane lattice in some Euclidean space so that all edges are represented by line segments of unit length; if n is even, each cylindrical $n \times n$ lattice is straightforwardly realized in (Euclidean) 3-space as a prism, while each toroidal $n \times n$ lattice is geometrically realized in a suitable duoprism in 4-space [9, 10]. Furthermore, in both cases a pair of generators is provided by any pair of centrally symmetric nodes (with respect to the origin).

5 CONCLUSION

In Section 2, the known notion of geodetic number acquires a new interpretation in logistics networks. In Section 3, the geodetic numbers are identified for the graphs of the five regular polyhedra and conditions are given that are sufficient to ensure that for the geodetic number of the graph of a polyhedron is equal to 2. As part of future research, it is worth (1) checking whether these conditions apply to the graphs of five regular polyhedra and (2) giving a description of the class of polytopes with geodesic number equal to 2. Regarding regular polyhedra, it would be interesting to find the total number of geodesics connecting N and S, and then determine how many of them are geometrically incongruent. In Section 4, the geodetic numbers are determined for the lattice graphs in the plane, cylinder, and the torus.

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