8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

On resolvent signless Laplacian Estrada index of graphs

Ramin Nasiri[1]
Department of Mathematics, Faculty of Sciences, Imam Hossein Comprehensive University, Tehran, Iran

Ali Nakhaei Amroodi
Department of Computer, network and communications, Imam Hossein Comprehensive University, Tehran, Iran

Abstract

For a simple non-complete graph G on n vertices, the resolvent signless Laplacian Estrada index of G is defined as $SLEE_r(G) = \sum_{i=1}^{n} \left(1 - \frac{q_i}{2n-2}\right)^{-1}$, where $q_1 \geq q_2 \geq \ldots \geq q_n$ are the eigenvalues of the signless Laplacian matrix of G. In this work, we introduce the extremal trees with respect to this consept, finally we establish some lower bounds for $SLEE_r$.

Keywords: resolvent signless Laplacian Estrada index, spectral moment, eigenvalues.

1 Introduction

For a simple graph G of order n, the adjacency matrix $\mathbf{A} = [a_{ij}]$ of G is the $n \times n$ matrix in which $a_{ij} = 1$ if the vertices i and j are adjacent, and $a_{ij} = 0$ otherwise. The matrix $\mathbf{Q}(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is called the signless Laplacian matrix of G, where $\mathbf{D}(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of G. The set of all eigenvalues of $\mathbf{Q}(G)$ are denoted by $\mathbf{Q}{-}Spec(G) = \{q_1, \ldots, q_n\}$, $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$.

It is well-known that $q_1 = 2\,n - 2$ if and only if G is a complete graph K_n. Hence, we have to consider non-complete graphs in definition of resolvent signless Laplacian Estrada index of G. The resolvent signless Laplacian Estrada index of G is defined as:

$$SLEE_r(G) = \sum_{i=1}^{n} \left(1 - \frac{q_i}{2n-2}\right)^{-1}. \tag{1}$$

One can see that we can write this quantity as:

$$SLEE_r(G) = tr\left(\mathbf{I} - \frac{1}{2n-2}\mathbf{Q}(G)\right)^{-1}. \tag{2}$$

Notice that if $G \ncong K_n$, then for each $i = 0, 1, \ldots, n$, $q_i < 2n-2$, and therefore $0 \leq \frac{q_i}{2n-2} < 1$. Thus, we may use the Maclaurin series for $(1 - \frac{q_i}{2n-2})^{-1}$ to evaluate $SLEE_r(G)$. In an exact phrase,

$$SLEE_r(G) = \sum_{k \geq 0} \frac{T_k(G)}{(2n-2)^k}, \tag{3}$$

[1]speaker

where $T_k(G)$ denotes to the $k-$th signless Laplacian spectral moment of the graph G, i.e. $T_k(G) = \sum_{i=1}^{n} q_i^k$. It is well-known that $T_k(G)$ equals to the number of closed semi-edge walks of length k, see [?] and so for some small values of k, it is possible to evaluate $T_k(G)$ in terms of some graph parameters. For example, $T_0(G) = n$, $T_1(G) = 2\,m$, $T_2(G) = Zg_1(G) + 2\,m$ and $T_3(G) = 6\,t + 3\,Zg_1(G) + \sum_{v \in V(G)} d^3(v)$, where $Zg_1(G)$ and $Zg_2(G)$ are the first and second Zagreb indices of garph G.

2 Main results

In this section, some results for the resolvent signless Laplacian Estrada index are presented. Assume that G is a simple n-vetex graph with signless Laplacian eigenvalues $q_1 \geq q_2 \geq \ldots \geq q_n$. In the following Lemma, the resolvent signless Laplacian Estrada index is computed by the characteristic polynomial of $\mathbf{Q}$.

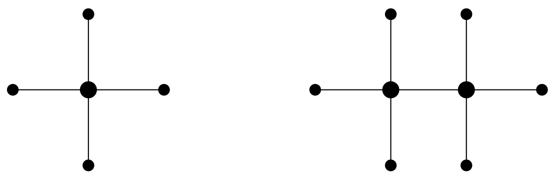
Lemma 2.1. *Let G be a non-complete $n-$vertex graph. Then,*

$$SLEE_r(G) = (2n-2)\frac{\Phi_G^{'}(2n-2)}{\Phi_G(2n-2)},$$

where $\Phi_G(x)$ is the characteristic polynomial of the matrix $\mathbf{Q}$.

In the following, the quantity $SLEE_r$ for some trees are computed. We start by an example applicable in chemistry.

Example 2.2. Apply Lemma **??**, we can calculate the resolvent signless Laplacian Estrada index of graphs by their $\mathbf{Q}-$polynomial. As an example, one can easily seen that $\Phi_{CH_4}(x) = x^5 - 8x^4 + 18x^3 - 16x^2 + 5x$ and $\Phi_{C_2H_6}(x) = x^8 - 14x^7 + 73x^6 - 182x^5 + 244x^4 - 182x^3 + 73x^2 - 14x + 1$, where CH_4 and C_2H_6, Figure 1, are molecular graphs of the Methane and Ethane molecules, respectively.



$C\,H_4$ $C_2\,H_6$

Figure 1: Molecular graphs of Methane and Ethane

Hence, by Lemma **??**,

$$SLEE_r(CH_4) = 8.\frac{5x^4 - 32x^3 + 54x^2 - 32x + 5}{x^5 - 8x^4 + 18x^3 - 16x^2 + 5x}\Big|_{x=8} \simeq 7.09524$$

$$SLEE_r(C_2H_6) = 14.\frac{8x^7 - 98x^6 + 438x^5 - 910x^4 + 976x^3 - 546x^2 + 146x - 14}{x^8 - 14x^7 + 73x^6 - 182x^5 + 244x^4 - 182x^3 + 73x^2 - 14x + 1}\Big|_{x=14} \simeq 9.37856.$$

Lemma 2.3. *Let G be a graph and $e \in E(G)$. Then $SLEE_r(G-e) < SLEE_r(G)$.*

Lemma 2.4. *Let T be a tree on n vertices, also P_n and S_n are the path and star trees, respectively. Then*

$$\sum_{i=1}^{n} \frac{n-1}{n-2-cos(\frac{\pi i}{n})} = SLEE_r(P_n) \leq SLEE_r(T) \leq SLEE_r(S_n) = \frac{2n^3 - 4n^2 - n + 4}{2n^2 - 7n + 6},$$

with left equality if and only if $T \cong P_n$ and right equality if and only if $T \cong S_n$.

Lemma 2.5. *Let G be a non-complete graph on n vertices. Then,*

$$n = SLEE_r(\overline{K_n}) \leq SLEE_r(G) \leq SLEE_r(K_n - e).$$

Moreover, if G is a connected non-complete graph on n vertices, then

$$SLEE_r(P_n) \leq SLEE_r(G) \leq SLEE_r(K_n - e).$$

Up to now, many lower and upper bounds for the largest and least signless Laplacian eigenvalues q_1 and q_n were given. In [?, ?], bounds on the signless Laplacian spectral radius $q_1(G)$ in terms of n and m of a connected graph G are investigated as:

$$\frac{4m}{n} \leq q_1(G) \leq \frac{2m}{n-1} + n - 2 \tag{4}$$

and left equality holds if and only if G is a regular graph and right equality holds if and only if G is S_n or K_n. Also, for the least $\mathbf{Q}-$eigenvalue $q_n(G)$ are the following [?, ?]:

$$\frac{2m}{n-2} - n + 1 \leq q_n(G) < \delta. \tag{5}$$

It is well known that the empty graph $\overline{K_n}$ is the unique graph with exactly one $\mathbf{Q}-$eigenvalue. Cvetković [?] proved that if G is a connected graph with r distinct signless Laplacian eigenvalues and diameter d, then $d \leq r - 1$. On the other hand, we know that the complete graph K_n is the unique connected graph with diameter one. Therefore, we can deduce that the complete graph K_n is the unique connected graph with exactly two $\mathbf{Q}-$eigenvalues. In the following Lemma, connected graphs which have three distinct $\mathbf{Q}-$eigenvalues are characterized.

Lemma 2.6. *[?] Let G be a connected graph of order $n \geq 4$. Then G has a Q-eigenvalue of multiplicity $n-2$ if and only if G is one of the graphs $K_n - e$, S_n, $K_{\frac{n}{2},\frac{n}{2}}$, $\overline{K_3 + S_4}$ or $\overline{K_1 + 2K_3}$.*

Example 2.7. In what follows, we compute the signless laplacian spectrum of graphs which are mentioned in the above lemma.

$$\begin{aligned}
&\mathbf{Q}-Spec(K_n - e) && = \{[\frac{3n-6 \pm \sqrt{n^2+4n-12}}{2}]^1, [n-2]^{n-2}\} \\
&\mathbf{Q}-Spec(S_n) && = \{[n]^1, [1]^{n-2}, [0]^1\} \\
&\mathbf{Q}-Spec(K_{\frac{n}{2},\frac{n}{2}}) && = \{[n]^1, [\frac{n}{2}]^{n-2}, [0]^1\} \\
&\mathbf{Q}-Spec(\overline{K_3 + S_4}) && = \{[9]^1, [4]^{n-2}, [1]^1\} \ ; n = 7 \\
&\mathbf{Q}-Spec(\overline{K_1 + 2K_3}) && = \{[9]^1, [4]^{n-2}, [1]^1\} \ ; n = 7
\end{aligned}$$

We are now ready to present some lower bounds for resolvent signless Laplacian Estrada index of graphs.

Lemma 2.8. *Let G be a graph with n vertices and m edges, and let $I \subseteq \{1, 2, \ldots, n\}$, then*

$$SLEE_r(G) \geq \sum_{j\in I} \frac{2n-2}{2n-2-q_j} + \frac{(2n-2)(n-n')^2}{(2n-2)(n-n') - 2m + \sum_{j\in I} q_j}$$

where $n' = n(I)$, and equality holds if and only if $q_i = q_j$, for all $i, j \notin I$.

Theorem 2.9. *Let G be a non-complete connected graph or empty graph with n vertices, m edges, maximum degree Δ, minimum degree δ and average degree $\overline{d}$. also let $1 \leq s < r \leq n$. Then,*

$$(1).\ SLEE_r(G) \geq \frac{n^2\,(n-1)}{n\,(n-1)-m}$$

$$(2).\ SLEE_r(G) \geq \frac{n\,(n-1)}{n\,(n-1)-2m} + \frac{n\,(n-1)^3}{n\,(n-1)^2+2m-mn}$$

$$(3).\ SLEE_r(G) \geq \frac{2\,n-2}{2\,n-2-q_2} + \frac{2\,n-2}{2\,n-2-q_s} + \frac{(2\,n-2)\,(n-2)^2}{(2\,n-2)(n-2)-2\,m+q_r+q_s}$$

$$(4).\ SLEE_r(G) > \frac{2\,n-2}{2\,n-2-2\,\overline{d}} + \frac{2\,n-2}{2\,n-1-\overline{d}} + \frac{(2\,n-2)\,(n-2)^2}{(2\,n-2)\,(n-2)-2\,m+n-2+2\,\Delta}$$
$$\geq \frac{2\,n-2}{2\,n-2-2\,\delta} + \frac{2\,n-2}{2\,n-1-\delta} + \frac{(2\,n-2)\,(n-2)^2}{(2\,n-2)\,(n-2)-2\,m+n-2+2\,\Delta}$$

The equalities in parts (1) and (2) hold if and only if $G \cong \overline{K_n}$, and in (3) holds if and only if $G \cong \overline{K_n}$, $K_n - e$, S_n, $K_{\frac{n}{2},\frac{n}{2}}$, $\overline{K_3 + S_4}$ or $\overline{K_1 + 2K_3}$.

Acknowledgement. We are very indebted to the referee for his/her suggestions and helpful remarks leaded us to rearrange the paper.

References

[1] F. Ayoobi, G. R. Omidi and B. Tayfeh-Rezaie, A note on graphs whose signless Laplacian has three distinct eigenvalues, *Linear Multilinear Algebra* **59** (6) (2011) 701–706.

[2] D. Cvetković, New theorems for signless Laplacians eigenvalues, *Bull. Cl. Sci. Math. Nat. Sci. Math.* **137** (33) (2008)

[3] D. Cvetković, P. Rowlinson and S. K. Simić, Signless Laplacians of finite graphs, *Linear Algebra Appl.* **423** (1) (2007) 155–171.

[4] D. Cvetković and S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian. III, *Appl. Anal. Discrete Math.* **4** (1) (2010) 156–166.

[5] K. Ch. Das, On conjectures involving second largest signless Laplacian eigenvalue of graphs, *Linear Algebra Appl.* **432** (11) (2010) 3018–3029.

[6] SG. Guo, YG. Chen and G. Yu, A lower bound on the least signless Laplacian eigenvalue of a graph, *Linear Algebra Appl.* **448** (2014) 217–221.

e-mail: `rnasiri@ihu.ac.ir`
e-mail: `Kpnakhaei@ihu.ac.ir`