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Some New Milne-Type Integral Inequalities for (h,m)-convex modified functions of second type on fractal sets

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Abstract

The main purpose of this paper is to obtain new versions of the Milne-Type Integral Inequalities for (h,m)-convex modified functions of second type on fractal sets. Several known results are derived from those obtained in our work.

Keywords: local fractional integrals, local fractional derivatives, fractal sets, Milne Inequality, (h,m)-convex modified functions of second type

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1 Introduction

A function $\varphi : [a_1, a_2] \to \mathbb{R}$ is said to be convex if $\varphi\big(\tau u + (1-\tau)v\big) \leq \tau\varphi(u) + (1-\tau)\varphi(v)$ holds for all $u, v \in [a_1, a_2]$ and $\tau \in [0,1]$. A function φ is said to be concave if $-\varphi$ is convex.

Convex functions have been generalized widely; highlighting the m-convex function, r-convex function, h-convex function, (h,m)-convex function, s-convex function and many others. Readers interested in going through many of these extensions and generalizations of the classical notion of convexity can consult [9].

Milne-type integral inequalities are a class of mathematical inequalities related to integrals. These inequalities are named after Edward Arthur Milne, a renowned mathematician known for his contributions to various areas of mathematics.

In general, a Milne-type integral inequality involves integrals of functions and provides bounds or inequalities for these integrals based on certain conditions or assumptions on the integrands and the integration domain. These inequalities are often used in mathematical analysis, particularly in integral calculus and related fields.

These inequalities are useful for estimating the magnitude of integrals in terms of other integrals, facilitating the analysis of various mathematical problems.

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Throughout the work we use the functions Γ (see [10, 11, 14, 15]).

The investigation and advancement of local fractional functions in fractal sets, including local fractional calculus and function continuity and monotonicity, are extensively studied in [13].

Following the above work, the real line number in fractal set $\mathbb{R}^\delta$ has the below properties.

If r_1^δ, r_2^δ, and $r_3^\delta \in \mathbb{R}^\delta$, $0 < \delta \leq 1$, then:

- $r_1^\delta + r_2^\delta \in \mathbb{R}^\delta$, $r_1^\delta r_2^\delta \in \mathbb{R}^\delta$.
- $r_1^\delta + r_2^\delta = r_2^\delta + r_1^\delta = (r_1 + r_2)^\delta = (r_2 + r_1)^\delta$.
- $r_1^\delta + (r_2^\delta + r_3^\delta) = (r_1^\delta + r_2^\delta) + r_3^\delta$.
- $r_1^\delta r_2^\delta = r_2^\delta r_1^\delta = (r_1 r_2)^\delta = (r_2 r_1)^\delta$.
- $(r_1^\delta r_2^\delta) r_3^\delta = r_1^\delta (r_2^\delta r_3^\delta)$.
- $r_1^\delta (r_2^\delta + r_3^\delta) = (r_1^\delta r_2^\delta) + (r_1^\delta r_2^\delta)$.
- $r_1^\delta + 0^\delta = 0^\delta + r_1^\delta = r_1^\delta$, and $r_1^\delta 1^\delta = 1^\delta r_1^\delta = r_1^\delta$.

Definition 1.1. The local fractional derivative of $f(x)$ of order δ at $x = x_0$ is given by

$$f^\delta(x_0) = \frac{d^\delta g}{dx^\delta}(x_0) = \lim_{x \to x_0} \frac{\Gamma(\delta+1)(g(x) - g(x_0))}{(x - x_0)^\delta}. \tag{1}$$

Definition 1.2. Let f be a local fractional continuous on $[a, b]$. The local fractional integral of $f(x)$ of order δ is given by

$${}_aI_b^\delta f(x) = \frac{1}{\Gamma(\delta+1)} \int_a^b f(t)(dt)^\delta. \tag{2}$$

Based on the previous Definition, we present the integral operators that we will use in our work.

Definition 1.3. Let f be a local fractional continuous on $[a, b]$. The right and left local fractional integral of f of order δ are given by

$${}^wI_{a+}^\delta f(b) = \frac{1}{\Gamma(\delta+1)} \int_a^b w^{(\delta)} \left(\frac{t-a}{b-a} \right) f(t)(dt)^\delta \tag{3}$$

and

$${}^wI_{b-}^\delta f(a) = \frac{1}{\Gamma(\delta+1)} \int_a^b w^{(\delta)} \left(\frac{b-t}{b-a} \right) f(t)(dt)^\delta \tag{4}$$

In this work we will use the following notion of convexity (which has as its starting point the definition of [2] and [5]).

Definition 1.4. Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\phi : I = [0, +\infty) \to \mathbb{R}^\alpha$. If inequality

$$\phi(\sigma\xi + m(1-\sigma)\varsigma) \leq h^\alpha(\sigma)\phi(\xi) + m^\alpha(1 - h^\alpha(\sigma))\phi(\varsigma) \tag{5}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\sigma \in [0, 1]$, where $m \in [0, 1]$, $s \in (0, 1]$. Then is said function ϕ is a generalized (h, m)-convex of first type on I.

Definition 1.5. Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\phi : I = [0,+\infty) \to \mathbb{R}^{\alpha}$. If inequality

$$\phi\left(\sigma\xi + m(1-\sigma)\varsigma\right) \leq h^{\alpha}(\sigma)\phi(\xi) + m^{\alpha}(1-h(\sigma))^{\alpha}\phi(\varsigma) \tag{6}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\sigma \in [0,1]$, where $m \in [0,1]$, $s \in (0,1]$. Then is said function ϕ is a generalized (h,m)-convex of second type on I.

Remark 1.6. Interested readers can easily verify that from Definition 1.5 we have many of the notions of convexity reported in the literature, for example, putting

- $h(z) = z$, $s = 1$, $m = 1$ and $\alpha = 1$, then ϕ is a convex function on $[0,+\infty)$ ([4, 9]).
- $h(z) = z$, $s = 1$ and $\alpha = 1$, then we have the m-convex of [12].
- $h(z) = z$, $m = 1$ and $\alpha = 1$, then we obtain the s-convex function on $[0,+\infty)$ ([3, 6]).
- $s = 1$ and $h(z) = z$, then we get Definition of generalized m-convex functions of [5].
- $m = s = 1$ and $h(z) = z$ we have the generalized convex function of [8].
- $m = 1$ and $h(z) = z$ we have the generalized s-convex function of [7].

Obviously, the reader will understand that under the consideration $\alpha = 1$, other known definitions of convexity can be reproduced.

In this work we obtain new variants of the classical Milne Inequality for functions generalized (h,m)-convex modified of second type, via local integral operators of the Definition 1.3.

2 Main results

As a first result, we get an equality which will be basic for obtaining the other results.

Lemma 2.1. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a local fractional differentiable function, with $\phi^{(\alpha)} \in L_1[a,b]$, $0 \leq a < b$. If $\frac{a}{m} \in [a,b]$, then we will have*

$$\begin{aligned}
& w(1)\left[\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{na+2b}{n+2}\right) - \left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{2a+nb}{n+2}\right)\right] \\
- \; & w(0)\left[\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{(n+1)a+b}{n+2}\right) - \left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{a+(n+1)b}{n+2}\right)\right] \\
- \; & \left[\left(\frac{n+2}{b-a}\right)^{2\alpha} {}^{w}J^{\alpha}_{\left(\frac{(n+1)a+b}{n+2}\right)+}\phi\left(\frac{na+2b}{n+2}\right) + \left(\frac{n+2}{b-a}\right)^{2\alpha} {}^{w}J^{\alpha}_{\left(\frac{a+(n+1)b}{n+2}\right)-}\phi\left(\frac{2a+nb}{n+2}\right)\right] \\
= \; & \int_0^1 w(t)\left[\phi^{(\alpha)}\left(\frac{n+1-t}{n+2}a + \frac{1+t}{n+2}b\right) - \phi^{(\alpha)}\left(\frac{1+t}{n+2}a + \frac{n+1-t}{n+2}b\right)\right] dt.
\end{aligned} \tag{7}$$

Proof. Let us denote

$$\begin{aligned}
&\int_0^1 w(t)\left[\phi^{(\alpha)}\left(\frac{n+1-t}{n+2}a+\frac{1+t}{n+2}b\right)-\phi^{(\alpha)}\left(\frac{1+t}{n+2}a+\frac{n+1-t}{n+2}b\right)\right]dt\\
=&\int_0^1 w(t)\phi^{(\alpha)}\left(\frac{n+1-t}{n+2}a+\frac{1+t}{n+2}b\right)dt\\
-&\int_0^1 w(t)\phi^{(\alpha)}\left(\frac{1+t}{n+2}a+\frac{n+1-t}{n+2}b\right)dt\\
=&\ I_1-I_2.
\end{aligned}$$

Integrating by parts in I_1 we have:

$$\begin{aligned}
I_1=&\left(\frac{n+2}{b-a}\right)^{\alpha}\left[w(1)\phi\left(\frac{na+2b}{n+2}\right)-w(0)\phi\left(\frac{(n+1)a+b}{n+2}\right)\right]\\
-&\left(\frac{n+2}{b-a}\right)^{\alpha}\int_0^1 w(t)\phi^{(\alpha)}\left(\frac{n+1-t}{n+2}a+\frac{1+t}{n+2}b\right)dt^{\alpha}.
\end{aligned}$$

Making a change of variables in this last integral and taking into account that $\frac{b-a}{n+2}=\frac{na+2b}{n+2}-\frac{(n+1)a+b}{n+2}$ we obtain:

$$\begin{aligned}
I_1=&\left(\frac{n+2}{b-a}\right)^{\alpha}\left[w(1)\phi\left(\frac{na+2b}{n+2}\right)-w(0)\phi\left(\frac{(n+1)a+b}{n+2}\right)\right]\\
-&\left(\frac{n+2}{b-a}\right)^{2\alpha}{}^{w}J^{\alpha}_{\left(\frac{(n+1)a+b}{n+2}\right)^+}\phi\left(\frac{na+2b}{n+2}\right).
\end{aligned}\tag{8}$$

In the same way you have to

$$\begin{aligned}
I_2=&\left(\frac{n+2}{b-a}\right)^{\alpha}\left[w(1)\phi\left(\frac{2a+nb}{n+2}\right)-w(0)\phi\left(\frac{a+(n+1)b}{n+2}\right)\right]\\
-&\left(\frac{n+2}{b-a}\right)^{2\alpha}{}^{w}J^{\alpha}_{\left(\frac{a+(n+1)b}{n+2}\right)^-}\phi\left(\frac{2a+nb}{n+2}\right).
\end{aligned}\tag{9}$$

Subtracting (9) from (8) and rearranging, we have the desired equality.
This ends the proof. □

Remark 2.2. In the case that $n=0$, $w(t)=(t+\frac{1}{3})^{\alpha}$, we have the Lemma 2.1 of [1] with $m=1$.

For convenience we will denote $L(\alpha,n,w)$ to left side of (7), so

$$\begin{aligned}
L(\alpha,n,w)=&\ w(1)\left[\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{na+2b}{n+2}\right)-\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{2a+nb}{n+2}\right)\right]\\
-&\ w(0)\left[\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{(n+1)a+b}{n+2}\right)-\left(\frac{n+2}{b-a}\right)^{\alpha}\phi\left(\frac{a+(n+1)b}{n+2}\right)\right]\\
-&\left[\left(\frac{n+2}{b-a}\right)^{2\alpha}{}^{w}J^{\alpha}_{\left(\frac{(n+1)a+b}{n+2}\right)^+}\phi\left(\frac{na+2b}{n+2}\right)+\left(\frac{n+2}{b-a}\right)^{2\alpha}{}^{w}J^{\alpha}_{\left(\frac{a+(n+1)b}{n+2}\right)^-}\phi\left(\frac{2a+nb}{n+2}\right)\right].
\end{aligned}\tag{10}$$

Theorem 2.3. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a local fractional differentiable (h,m)-convex modified function of second type, $m \in (0,1]$, with $\phi^{(\alpha)} \in L_1[a,b]$, $0 \le a < b$. If $\frac{a}{m} \in [a,b]$, then*

$$
\begin{aligned}
&|L(\alpha,n,w)| \le (w(1)-w(0))\phi\left(\frac{x+y}{2}\right)\\
\le\ & h^\alpha\left(\frac{1}{2}\right)\left(\frac{m(n+1)}{y-x}\right)^\alpha {}^wJ^\alpha_{\frac{(n+1)x+ny}{n+2}+}\phi\left(\frac{1}{m}\frac{nx+(n+1)y}{n+2}\right)\\
+\ & m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\left(\frac{n+2}{y-x}\right)^\alpha {}^wJ^\alpha_{\frac{nx+(n+1)y}{n+2}-}\phi\left(\frac{(n+1)x+ny}{n+2}\right)\\
\le\ & h^\alpha\left(\frac{1}{2}\right)\phi(x)\int_0^1 D^\alpha w(t)h^\alpha\left(\frac{n+1+t}{n+2}\right)dt^\alpha\\
+\ & m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\phi\left(\frac{y}{m}\right)\int_0^1 D^\alpha w(t)\left(1-h\left(\frac{1+t}{n+2}\right)\right)^\alpha dt^\alpha\\
+\ & h^\alpha\left(\frac{1}{2}\right)\phi\left(\frac{x}{m}\right)\int_0^1 D^\alpha w(t)h^\alpha\left(\frac{1+t}{n+2}\right)dt^\alpha\\
+\ & m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\phi\left(\frac{y}{m^2}\right)\int_0^1 D^\alpha w(t)\left(1-h\left(\frac{n+1+t}{n+2}\right)\right)^\alpha dt^\alpha. \qquad (11)
\end{aligned}
$$

Proof. Putting $t=\frac{1}{2}$ we have from the generalized (h,m)-convexity of ϕ the following

$$\phi\left(\frac{a+b}{2}\right) \le h^\alpha\left(\frac{1}{2}\right)\phi(a) + m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\phi\left(\frac{b}{m}\right).$$

Putting $a=\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y$ and $b=\frac{n+t}{n+2}x+\frac{n+1-t}{n+2}y$, we have

$$\phi\left(\frac{x+y}{2}\right) \le h^\alpha\left(\frac{1}{2}\right)\phi\left(\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y\right) + m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\phi\left(\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}\right).$$

Multiplying this inequality by $D^\alpha w(t)$ and integrating between 0 and 1 with respect to t we obtain

$$
\begin{aligned}
&\int_0^1 D^\alpha w(t)\phi\left(\frac{x+y}{2}\right)dt^\alpha \le h^\alpha\left(\frac{1}{2}\right)\int_0^1 D^\alpha w(t)\phi\left(\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y\right)dt^\alpha\\
+\ & m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha\int_0^1 D^\alpha w(t)\phi\left(\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}\right)dt^\alpha.
\end{aligned}
$$

From this we have

$$(w(1)-w(0))\phi\left(\frac{x+y}{2}\right) \le h^\alpha\left(\frac{1}{2}\right)I_1 + m^\alpha\left(1-h\left(\frac{1}{2}\right)\right)^\alpha I_2 \qquad (12)$$

Changing the variables $z=\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y$ in I_1 and $z=\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}$ in I_2 leads us to the following result

$$
\begin{aligned}
I_1 &= \int_0^1 D^\alpha w(t)\phi\left(\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y\right)dt^\alpha\\
&= \left(\frac{m(n+1)}{y-x}\right)^\alpha\int_{\frac{1}{m}\left(\frac{(n+1)x+ny}{n+2}\right)}^{\frac{1}{m}\left(\frac{nx+(n+1)y}{n+2}\right)} D^\alpha w\left(\frac{mz-\frac{(n+1)x+ny}{n+2}}{\frac{1}{m}\frac{ny-x}{n+2}}\right)\phi(z)dz^\alpha\\
&= \left(\frac{m(n+1)}{y-x}\right)^\alpha {}^wJ^\alpha_{\frac{(n+1)x+ny}{n+2}+}\phi\left(\frac{1}{m}\frac{nx+(n+1)y}{n+2}\right)
\end{aligned}
$$

Analogously for I_2 we have

$$I_2=\left(\frac{n+2}{y-x}\right)^{\alpha} {}^{w}J^{\alpha}_{\frac{nx+(n+1)y}{n+2}-}\phi\left(\frac{(n+1)x+ny}{n+2}\right).$$

Taking into account the last two results in (12), we can the first inequality of (11).

To obtain the right member, using the generalized (h,m)-convexity of ϕ we have sucesively

$$\phi\left(\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y\right)\leq h^{\alpha}\left(\frac{n+1+t}{n+2}\right)\phi(x)+m^{\alpha}\left(1-h\left(\frac{1+t}{n+2}\right)\right)^{\alpha}\phi\left(\frac{y}{m}\right)$$
$$\phi\left(\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}\right)\leq h^{\alpha}\left(\frac{1+t}{n+2}\right)\phi(x)+m^{\alpha}\left(1-h\left(\frac{n+1+t}{n+2}\right)\right)^{\alpha}\phi\left(\frac{y}{m^2}\right).$$

Multiplying the first inequality by $h^{\alpha}\left(\frac{1}{2}\right)D^{\alpha}w(t)$ and the second by $m^{\alpha}\left(1-h\left(\frac{1}{2}\right)\right)^{\alpha}D^{\alpha}w(t)$, after integrating between 0 and 1 we obtain

$$\begin{aligned}
&h^{\alpha}\left(\frac{1}{2}\right)\int_0^1 D^{\alpha}w(t)\phi\left(\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y\right)dt^{\alpha}\\
\leq\ & h^{\alpha}\left(\frac{1}{2}\right)\phi(x)\int_0^1 D^{\alpha}w(t)h^{\alpha}\left(\frac{n+1+t}{n+2}\right)dt^{\alpha}\\
+\ & m^{\alpha}h^{\alpha}\left(\frac{1}{2}\right)\phi\left(\frac{y}{m}\right)\int_0^1 D^{\alpha}w(t)\left(1-h\left(\frac{1+t}{n+2}\right)\right)^{\alpha}dt^{\alpha}
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\left(1-h\left(\frac{1}{2}\right)\right)^{\alpha}\int_0^1 D^{\alpha}w(t)\phi\left(\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}\right)dt^{\alpha}\\
\leq\ & h^{\alpha}\left(\frac{1}{2}\right)\phi\left(\frac{x}{m}\right)\int_0^1 D^{\alpha}w(t)h^{\alpha}\left(\frac{1+t}{n+2}\right)dt^{\alpha}\\
+\ & m^{\alpha}\left(1-h\left(\frac{1}{2}\right)\right)^{\alpha}\phi\left(\frac{y}{m^2}\right)\int_0^1 D^{\alpha}w(t)\left(1-h\left(\frac{n+1+t}{n+2}\right)\right)^{\alpha}dt^{\alpha}
\end{aligned}\tag{14}$$

Making the change of variables $z=\frac{n+1-t}{n+2}x+\frac{1+t}{n+2}y$ in the integral of the left side of (13) and $z=\frac{1+t}{n+2}\frac{x}{m}+\frac{n+1-t}{n+2}\frac{y}{m}$ in the integral of the left side of (14), the required inequality is obtained.

This ends the proof. □

Remark 2.4. If in the above result we put $h(t)=t$, $n=0$ and $D^{\alpha}w(t)=1$, we obtain a extension of the Theorem 3.1 of [5].

By imposing more restrictive conditions on ϕ^{α} on the right side of (7), we can obtain more refined inequalities.

So, we have this first result.

Theorem 2.5. *Let $\phi:[0,\infty)\to\mathbb{R}^{\alpha}$, with $\phi^{(\alpha)}$ be a (h,m)-convex modified function of second type, $m\in(0,1]$, with $\phi^{(\alpha)}\in L_1[a,b]$, $0\leq a<b$. If $\frac{a}{m}\in[a,b]$, then*

$$\begin{aligned}
&|L(\alpha,n,w)|\\
\leq\ & \left|\phi^{(\alpha)}(a)\right|\left[\int_0^1 w(t)\left(h^{\alpha}\left(\frac{n+1-t}{n+2}\right)+h^{\alpha}\left(\frac{1+t}{n+2}\right)\right)dt^{\alpha}\right]\\
+\ & m^{\alpha}\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\left[\int_0^1 w(t)\left(\left(1-h\left(\frac{1+t}{n+2}\right)\right)^{\alpha}+\left(1-h\left(\frac{n+1-t}{n+2}\right)\right)^{\alpha}\right)dt^{\alpha}\right].
\end{aligned}\tag{15}$$

Proof. From Lemma 2.1 and using properties of fractal integral we have

$$\begin{aligned}&\left|\int_0^1 w(t)\left[\phi^{(\alpha)}\left(\frac{n+1-t}{n+2}a+\frac{1+t}{n+2}b\right)-\phi^{(\alpha)}\left(\frac{1+t}{n+2}a+\frac{n+1-t}{n+2}b\right)\right]dt^\alpha\right|\\ \leq\ & |I_1|+|I_2|.\end{aligned}$$

And using the (h,m)-convexity of $\phi^{(\alpha)}$ in both integrals, leads us to

$$\begin{aligned}&|I_1|+|I_2|\\ \leq\ & \left|\phi^{(\alpha)}(a)\right|\left[\int_0^1 w(t)\left(h^\alpha\left(\frac{n+1-t}{n+2}\right)+h^\alpha\left(\frac{1+t}{n+2}\right)\right)dt^\alpha\right]\\ +\ & m^\alpha\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\left[\int_0^1 w(t)\left(\left(1-h\left(\frac{1+t}{n+2}\right)\right)^\alpha+\left(1-h\left(\frac{n+1-t}{n+2}\right)\right)^\alpha\right)dt^\alpha\right].\end{aligned}$$

Which is the desired result. □

Theorem 2.6. *Under assumptions of previous result if $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&|L(\alpha,n,w)|\\ \leq\ & \left(\int_0^1 w^p(t)dt^\alpha\right)^{\frac{1}{p}}\left\{\left(\left|\phi^{(\alpha)}(a)\right|\int_0^1 h^\alpha\left(\frac{n+1-t}{n+2}\right)dt^\alpha+m^\alpha\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\int_0^1 h^\alpha\left(\frac{1+t}{n+2}\right)dt^\alpha\right)^{\frac{1}{q}}\right.\\ +\ & \left.\left(\left|\phi^{(\alpha)}(a)\right|\int_0^1 h^\alpha\left(\frac{1+t}{n+2}\right)dt^\alpha+m^\alpha\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\int_0^1 h^\alpha\left(\frac{n+1-t}{n+2}\right)dt^\alpha\right)^{\frac{1}{q}}\right\}. \qquad (16)\end{aligned}$$

Proof. Using Lemma 2.1, the (h,m)-convexity of $\phi^{(\alpha)}$ and the well konwn Hölder inequality, on the member right, the desired result is obtained. □

Theorem 2.7. *Under assumptions of Theorem 2.5 if $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality*

$$\begin{aligned}&|L(\alpha,n,w)|\\ \leq\ & \left(\int_0^1 w(t)dt^\alpha\right)^{1-\frac{1}{q}}\left\{\left(\left|\phi^{(\alpha)}(a)\right|\int_0^1 w(t)h^\alpha\left(\frac{n+1-t}{n+2}\right)dt^\alpha+m^\alpha\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\int_0^1 w(t)h^\alpha\left(\frac{1+t}{n+2}\right)dt^\alpha\right)^{\frac{1}{q}}\right.\\ +\ & \left.\left(\left|\phi^{(\alpha)}(a)\right|\int_0^1 w(t)h^\alpha\left(\frac{1+t}{n+2}\right)dt^\alpha+m^\alpha\left|\phi^{(\alpha)}\left(\frac{b}{m}\right)\right|\int_0^1 w(t)h^\alpha\left(\frac{n+1-t}{n+2}\right)dt^\alpha\right)^{\frac{1}{q}}\right\}. \qquad (17)\end{aligned}$$

Proof. The proof follows a similar path to the previous one, although Mean Power Inequality is used instead of Hölder's Inequality. □

Remark 2.8. Other refinements can be obtained using other known inequalities, such as Young's.

3 Conclusions

In this work we present a generalized formulation of the fractal weighted integral, which contains as particular cases many of the integral operators reported in the literature. In this context, we present several integral inequalities that generalize several known inequalities.

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