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Equivalence scheme to prove the existence and uniqueness solution of fractional differential equations

K. Sayevand*, F. Shahsavand

Faculty of Mathematical Sciences, Malayer University, Malayer, Iran

ksayevand@yahoo.com, shahsavand4@gmail.com

ABSTRACT

In this study, we give some extensions of the fundamental existence and uniqueness theorems for ordinary fractional differential equation based on the reducing Cauchy type problems. The obtained results show that, proof is shorter and easier.

KEYWORDS: Fractional calculus, equivalency scheme, existence, uniqueness.

1 INTRODUCTION

Fractional calculus appears naturally in various fields of mathematics [1–3]. During the last decade, fractional calculus was found to play a fundamental role in the modeling of a considerable number of phenomena, in particular, the modeling of memory dependent phenomena and complex media such as porous media. Most of the investigations in this field involve the existence and uniqueness of solutions to fractional differential equations with the Riemann-Liouville fractional derivative $D^{\alpha}_{t_0+}u$ defined for $(\mathrm{R}(\alpha) > 0)$.

Definition 1 ([4]). Let $\Omega = [t_0, t_1]$ $(-\infty < \mathrm{t}_0 < \mathrm{t}_1 < \infty)$ be a finite interval on the real axis R . The Riemann-Liouville fractional integrals $L^{\alpha}_{t_0+} f$ and $L^{\alpha}_{t_0-} f$ of order $\alpha \in C$ $(\mathrm{R}(\alpha) > 0)$ are defined by

$$(L^{\alpha}_{t_0+} f)(\mathrm{t}) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\mathrm{x})dx}{(\mathrm{t}-\mathrm{x})^{1-\alpha}}, \quad \mathrm{t} > \mathrm{t}_0, \mathrm{R}(\alpha) > 0, \tag{1}$$

and

$$(L^{\alpha}_{t_n-} f)(\mathrm{t}) = \frac{1}{\Gamma(\alpha)} \int_t^{t_n} \frac{f(\mathrm{x})dx}{(\mathrm{t-x})^{1-\alpha}}, \quad \mathrm{t} > \mathrm{t}_n, \mathrm{R}(\alpha) > 0. \tag{2}$$

The Riemann-Liouville fractional derivatives $D^{\alpha}_{t_0+}u$ and $D^{\alpha}_{t_n-}u$ of order $\alpha \in C$ $(\mathrm{R}(\alpha) \geq 0)$ are defined by

$$(D^{\alpha}_{t_0+}u)(\mathrm{t}) = (\frac{d}{dt})^n (\mathrm{L}^{n-\alpha}_{t_0+} u)(\mathrm{t}) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} (\frac{d}{dt})^n \int_{t_0}^{t} \frac{u(\mathrm{x})dx}{(\mathrm{t-x})^{\alpha-n+1}}, \quad n = [\mathrm{R}(\alpha)]+1, t > t_0, \tag{3}$$

and

$$(D^{\alpha}_{t_0-}u)(\mathrm{t}) = (-\frac{d}{dt})^n (\mathrm{L}^{n-\alpha}_{t_0-} u)(\mathrm{t}) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} (-\frac{d}{dt})^n \int_{t}^{t_n} \frac{u(\mathrm{x})dx}{(\mathrm{t-x})^{\alpha-n+1}}, \quad n = [\mathrm{R}(\alpha)]+1, t > t_n. \tag{4}$$

Assume that the nonlinear differential equation of fractional order $\alpha \in C$ $(\mathrm{R}(\alpha) > 0)$ on a finite interval $[\mathrm{t}_0, \mathrm{t}_n] \in R = (-\infty, \infty)$ has the form

$$(D^{\alpha}_{t_0-}(\mathrm{u} - \mathrm{T}_{m-1}[\mathrm{u}]))(\mathrm{t}) = \mathrm{f}[\mathrm{t}, \mathrm{u}(\mathrm{t})], \quad \mathrm{R}(\alpha) > 0, t > t_0, \tag{5}$$

where T_{m-1} is Taylor expansion of order $[\alpha] = \mathrm{m}$ and with initial conditions

$$(D^{\alpha-m}_{t_0+}(\mathrm{u} - \mathrm{T}_{m-1}[\mathrm{u}]))(\mathrm{t}) = l_m, \quad l_m \in C, \quad m = 1, 2, ..., \mathrm{n}, \tag{6}$$

where

$$\begin{cases} n = R(\alpha)+1 & for \quad \alpha \notin N, \\ \alpha = n & for \quad \alpha \in N, \end{cases}$$

limits at almost all points of the right-sided neighborhood $(\mathrm{t}_0, \mathrm{t}_0 + \varepsilon)(\varepsilon > 0)$ shows by

$(D^{\alpha-m}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}_0 +)$ as follows:

$$(D^{\alpha-m}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}_0 +) = \lim_{t \to t_0+} (D^{\alpha-m}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}), \quad (1 \leq \mathrm{m} \leq \mathrm{n}-1), \tag{7}$$

and

$$\begin{cases} (D^{\alpha-n}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}_0 +) = \lim_{t \to t_0+} (\mathrm{L}^{n-\alpha}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}), & \text{for } \alpha \neq \mathrm{n}, \\ (D^{0}_{t_0+}(u - T_{m-1}[\mathrm{u}]))(\mathrm{t}_0 +) = \mathrm{u}(\mathrm{t}_0), & \text{for } \alpha = \mathrm{n}. \end{cases} \tag{8}$$

If $\alpha = n \in N$, then the problem in (5)-(6) is reduced to the usual Cauchy problem for the ordinary differential equation of order $n \in N$:

$$\begin{cases}(u-T_{m-1}[\mathrm{u}])^{(\mathrm{n})}(\mathrm{t})=\mathrm{f}\ [\mathrm{t},\mathrm{u}(\mathrm{t})],\\ \mathrm{u}^{(\mathrm{n-m})}(\mathrm{t}_0)=l_m,\ l_m\in C,\ \ \mathrm{m}=1,...,\mathrm{n}.\end{cases} \tag{9}$$

The problem (5)-(6) is therefore called, by analogy, a Cauchy type problem.When $0<R(\alpha)<1$, the problem (5)-(6) takes the form

$$(D^{\alpha}_{t_0+}(u-T_{m-1}[\mathrm{u}]))(\mathrm{t})=\mathrm{f}[\mathrm{t},\mathrm{u}(\mathrm{t})],\ \ (D^{1-\alpha}_{t_0+}\ \mathrm{u})(\mathrm{t}_0+)=\mathrm{b},\mathrm{b}\in C. \tag{10}$$

Essentially, they were based on reducing problem (5)-(6) to the following nonlinear Volterra integral equation of the second kind:

$$(u-T_{m-1}[\mathrm{u}])(\mathrm{t})=\sum_{j=1}^{n}\frac{l_j}{\Gamma(\alpha-\mathrm{j}+1)}(\mathrm{t}-\mathrm{t}_0)^{\alpha-j}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f[v,\mathrm{u}(v)]dv}{(\mathrm{t}-v)^{1-\alpha}},\ \ \mathrm{t}>\mathrm{t}_0. \tag{11}$$

Definition 2. Let $[t_0,t_1]\ (-\infty<\mathrm{t}_0<\mathrm{t}_n<\infty)$ be a finite interval and $AC[\mathrm{t}_0,\mathrm{t}_n]$ be the space of functions f which are absolutely continuous on $[t_0,t_1]$. It is known that $AC[\mathrm{t}_0,\mathrm{t}_n]$ coincides with the space of primitives of Lebesgue sum able functions:

$$f\ (\mathrm{t})\in AC[\mathrm{t}_0,\mathrm{t}_n]\Leftrightarrow f\ (\mathrm{t})=\mathrm{c}+\int_{t_0}^{t}\eta(\mathrm{x})dx\ (\eta(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)),$$

$$\forall n\in N:AC^n[\mathrm{t}_0,\mathrm{t}_n]=\left\{\ f:[\mathrm{t}_0,\mathrm{t}_n]\to C\ \ \ and\ \ \ [(\mathrm{D}^{n-1}\mathrm{f})(\mathrm{t})]\in AC[\mathrm{t}_0,\mathrm{t}_n]\right\}.$$

Lemma 1 ([4]). The space $AC^n[\mathrm{t}_0,\mathrm{t}_n]$ consists of those and only those functions $f\ (\mathrm{x})$ which can be represented in the form

$$\mathrm{f}(\mathrm{t})=(\mathrm{L}^n_{t_0})\eta(\mathrm{t})+\sum_{k=0}^{n-1}c_k\,(\mathrm{t}-\mathrm{t}_0)^k, \tag{١٢}$$

where $\eta(\mathrm{t})\in\mathrm{L}(\mathrm{t}_0,\mathrm{t}_n),\mathrm{c}_k\ (\mathrm{k}=0,1,...,\mathrm{n}-1)$ are arbitrary constants, and

$$(\mathrm{L}^n_{t_0+}\,\eta)(\mathrm{t})=\frac{1}{(\mathrm{n}-1)!}\int_{t_0}^{t}(\mathrm{t}-\mathrm{x})^{n-1}\eta(\mathrm{x})dx. \tag{١٣}$$

Lemma 2 ([4]). If $\mathrm{R}>0$ and $f\ (\mathrm{x})\in\mathrm{L}_p(\mathrm{t}_0,\mathrm{t}_n)(1\le\mathrm{p}\le\infty)$, then

$$\forall\ \mathrm{t}\in[\mathrm{t}_0,\mathrm{t}_n],\ \ (\mathrm{D}^{\alpha}_{t_0+}\,L^{\alpha}_{t_0+}f)(\mathrm{t})=(\mathrm{D}^{\alpha}_{t_n-}\,L^{\alpha}_{t_n-}f)(\mathrm{t})=\mathrm{f}(\mathrm{t}),(\mathrm{R}(\alpha)>0). \tag{14}$$

Lemma 3 ([4]). Let $\mathrm{R}>0,\mathrm{n}=[\mathrm{R}]+1$ and $f_{n-\alpha}(\mathrm{x})=(L^{\alpha}_{t_0+}f)(\mathrm{x})$ be the fractional integral (1) of order $\mathrm{n}-\alpha$

a) If $1 \le p \le \infty$ and $f\,(\mathrm{t}) \in L^{\alpha}_{t_0+}(\mathrm{L}_p)$, then

$$(L^{\alpha}_{t_0+} D^{\alpha}_{t_0+} f\,)(\mathrm{t}) = f\,(\mathrm{t}). \tag{١٥}$$

b) If $f\,(\mathrm{x}) \in L_1(\mathrm{t}_0, \mathrm{t}_n)$ and $f_{n-\alpha}(\mathrm{x}) \in \mathrm{AC}^n[\mathrm{t}_0, \mathrm{t}_n]$, then

$$\forall\, \mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_n]\ (L^{\alpha}_{t_0+} D^{\alpha}_{t_0+} f)(\mathrm{t}) = \mathrm{f(t)} - \sum_{j=1}^{n} \frac{f^{(\mathrm{n-j})}_{n-\alpha}(\mathrm{t}_0)}{\Gamma(\alpha - j + 1)} (\mathrm{t} - \mathrm{t}_0)^{\alpha}. \tag{16}$$

2 EQUATIONS WITH THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE IN THE SPACE OF SUMMABLE FUNCTIONS

In this section, we give conditions for a unique global solution to the Cauchy type problem (5)-(6) in the space $L^{\alpha}(\mathrm{t}_0, \mathrm{t}_n)$ defined for $\alpha \in C(\mathrm{R}(\alpha) > 0)$ by

$$\mathrm{L}^{\alpha}(\mathrm{t}_0, \mathrm{t}_n) = \left\{ u \in L(\mathrm{t}_0, \mathrm{t}_n); \quad D^{\alpha}_{t_0+}(\mathrm{u} - \mathrm{T}_{m-1}[\mathrm{u}]) \in \mathrm{L}(\mathrm{t}_0, \mathrm{t}_n) \right\}. \tag{17}$$

Here, $L(\mathrm{t}_0, \mathrm{t}_n) = L_1(\mathrm{t}_0, \mathrm{t}_n)$ is the space of sum able functions in a finite interval $[\mathrm{t}_0, \mathrm{t}_n]$ of the real axis R. We also generalize the obtained result to the Cauchy type problem for fractional differential equations more general than (5) and to the system Cauchy type problem (5)-(6). Our investigations are based on reducing the problems considered to Volterra integral equations of the second kind and on using the Banach fixed point theorem.

2.1 Equivalence of the Cauchy type problem and the Volterra integral equation

Hereunder, we prove that the Cauchy type problem (5)-(6) and the nonlinear Volterra integral equation (11) are equivalent in the sense that, if $\mathrm{u} - \mathrm{T}_{m-1}[\mathrm{u}] \in \mathrm{L}(\mathrm{t}_0, \mathrm{t}_n)$ satisfies one of these relations, then it also satisfies the other. We prove such a result by assuming that a function $f\,[\mathrm{t}, \mathrm{u}]$ belongs to $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_n)$ for any $\mathrm{u} \in \mathrm{G} \subset C$. For this we need the auxiliary assertion following Lemma.

Lemma 4 ([4]). The fractional integration operator $L^{\alpha}_{t_0+}$ (wherein $\alpha \in C \quad R(\alpha) > 0$) is bounded in $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_n)$:

$$\left\| L^{\alpha}_{t_0+} g \right\|_1 \le \frac{(\mathrm{t}_n - \mathrm{t}_0)^{R(\alpha)}}{R(\alpha) \left| \Gamma(\alpha) \right|} \left\| g \right\|_1, \tag{١٨}$$

also, for $\alpha > 0$, the estimate (18) takes the form

$$\left\| L^{\alpha}_{t_0+} g \right\|_1 \le \frac{(\mathrm{t}_n - \mathrm{t}_0)^{\alpha}}{\Gamma(\alpha + 1)} \left\| g \right\|_1. \tag{١٩}$$

Again, consider the Cauchy type problem (5)-(6) with real $\alpha > 0$:

$$(D_{t_0+}^{\alpha}(\mathrm{u}-\mathrm{T}_{m-1}[\mathrm{u}]))(\mathrm{t})= f[\mathrm{t},\mathrm{u}(\mathrm{t})]\ \ (\alpha>0), \tag{20}$$

$$(D_{t_0+}^{\alpha-m}(u-T_{m-1}[\mathrm{u}]))(\mathrm{t}_0+)=l_m,\ \ l_m\in R,\ \ \mathrm{m}=1,...,\mathrm{n}=-[-\alpha]. \tag{21}$$

Now, based on these equations we present the following results:

Theorem 1. Let $\alpha>0$, $\mathrm{n}=-[-\alpha]$. Let $G\subset R$ be an open and $f:(\mathrm{t}_0,\mathrm{t}_n]\times G\to R$ be a function such that $\forall u\in G, u(\mathrm{t})\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)\Rightarrow f[\mathrm{t},u]\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$. Consequently $u(\mathrm{t})$ satisfies the relations (20) and (21) if and only if $u(\mathrm{t})$ satisfies the integral equation (11).

Proof. Let $u(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$ satisfy the relations (20) and (21). Since $f[\mathrm{t},u]\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$, (20) means that there exists on $[\mathrm{t}_0,\mathrm{t}_n]$ the fractional derivative $(D_{t_0+}^{\alpha-m}u)(\mathrm{t})\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$. Therefore

$$(D_{t_0+}^{\alpha}(u-T_{m-1}[u]))(\mathrm{t})=(\frac{d}{dt})^n(\mathrm{L}_{t_0+}^{n-\alpha})(-\mathrm{T}_{m-1}[u])(\mathrm{t}),\ \ \mathrm{n}=-[-\alpha],\ \ (\mathrm{L}_{t_0+}^{0}u)(\mathrm{t})=u(\mathrm{t}), \tag{22}$$

and hence, by Lemma 1, $(\mathrm{L}_{t_0+}^{n-\alpha}(u-T_{m-1}[\mathrm{u}]))(\mathrm{t})\in \mathrm{AC}^n[\mathrm{t}_0,\mathrm{t}_n]$. Thus we can apply Lemma 3, we have

$$(\mathrm{L}_{t_0+}^{\alpha}D_{t_0+}^{\alpha})(u-T_{m-1}[\mathrm{u}])(\mathrm{t})=\mathrm{u}(\mathrm{t})\sum_{j=1}^{n}\frac{u_{n-\alpha}^{(\mathrm{n-j})}(\mathrm{t}_0)}{\Gamma(\alpha-\mathrm{j}+1)}(\mathrm{t}-\mathrm{t}_0)^{\alpha-j},\ \ u_{n-\alpha}(\mathrm{t})=(\mathrm{L}_{t_0+}^{n-\alpha}\mathrm{u})(\mathrm{t}). \tag{23}$$

From (29), we have

$$u_{n-\alpha}^{(\mathrm{n-j})}(\mathrm{t})=(\frac{d}{dt})^{n-j}(\mathrm{L}_{t_0+}^{(\mathrm{n-j})-(\alpha-\mathrm{j})}u)(\mathrm{t}). \tag{۲۴}$$

From (24) and (9), we rewrite (23) in the form

$$(L_{t_0+}^{\alpha}D_{t_0+}^{\alpha}(u-T_{m-1}[u]))(\mathrm{t})=u(\mathrm{t})-\sum_{j=1}^{n}\frac{(D_{t_0+}^{\alpha-j}u)(\mathrm{t}_0+)}{\Gamma(\alpha-\mathrm{j}+1)}(\mathrm{t}-\mathrm{t}_0)^{\alpha-j}=u(\mathrm{t})-\sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-\mathrm{j}+1)}(\mathrm{t}-\mathrm{t}_0)^{\alpha-j}, \tag{25}$$

By Lemma 4, the integral $(\mathrm{L}_{t_0+}^{\alpha}f[\mathrm{t},u(\mathrm{t})])(\mathrm{t})\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$ exists on $[\mathrm{t}_0,\mathrm{t}_n]$. Applying the operator $\mathrm{L}_{t_0+}^{\alpha}$ to both sides of (8) and using (25) and (1), we obtain the equation (11). Let $u(\mathrm{t})\in \mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$ satisfy the equation (11). Applying the operator $D_{t_0}^{\alpha}$ to both sides of (11), we have

$$(\mathrm{D}_{t_0+}^{\alpha}(\mathrm{u}-\mathrm{T}_{m-1}[\mathrm{u}]))(\mathrm{t})=\sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-\mathrm{j}+1)}(\mathrm{D}_{t_0+}^{\alpha}(x-\mathrm{t}_0)^{\alpha-j})(\mathrm{t})+(\mathrm{D}_{t_0+}^{\alpha}L_{t_0+}^{\alpha}f[\mathrm{t},\mathrm{u}(\mathrm{t})](\mathrm{t}).$$

From here, in accordance with the Lemma 2 (with $f(\mathrm{t})$ replaced by $f[\mathrm{t},\mathrm{u}(\mathrm{t})]$), we arrive at the equation (20). Now we show that the relations in (21) also hold. For this we apply the operators $\mathrm{D}_{t_0+}^{\alpha-m}(m=1,...,n)$ to both sides of (11). If $1\le m\le n-1$, then we have

$$(D_{t_0+}^{\alpha-m}(u-T_{m-1}[u]))(t)=\sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-j+1)}(D_{t_0+}^{\alpha-m}(x-t_0)^{\alpha-j})(t)+(D_{t_0+}^{\alpha-m}L_{t_0+}^{\alpha}\ f[t,u(t)](t)$$
$$=\sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-j+1)}(t-t_0+)^{m-j}+L_{t_0+}^{m}\ f[t,u(t)](t).$$

Therefore

$$(D_{t_0+}^{\alpha-m}(u-T_{m-1}[u]))(t)=\sum_{j=1}^{n}\frac{b_j}{(m-j)!}(t-t_0)^{m-j}+\frac{1}{(m-1)!}\int_{t_0}^{t}(t-x)^{m-1}f\,[t,u(t)]dx. \qquad (26)$$

If $m=n$, then, in accordance with (8) and similarly to (26), we obtain

$$(D_{t_0+}^{\alpha-n}(u-T_{m-1}[u]))(t)=\sum_{j=1}^{n}\frac{b_j}{(n-j)!}(t-t_0)^{m-j}+\frac{1}{(n-1)!}\int_{t_0}^{t}(t-x)^{n-1}f\,[t,u(t)]dx. \qquad (٢٧)$$

Taking in (26) and (27) a limit as $x\to t_0+$, we obtain the relations in (21). Thus the sufficiency is proved, which completes the proof of Theorem 1.

Corollary 1. If $(u-T_{m-1}[u]))(t)\in L(t_0,t_n)$, then $(u-T_{m-1}[u]))(t)$ satisfies the relations in (9) with $l_m\in R(m=1,...,n)$ if, and only if, $(u-T_{m-1}[u]))(t)$ satisfies the integral equation

$$(u-T_{m-1}[u])(t)=\sum_{j=1}^{n}\frac{b_j}{(n-j)!}(t-t_0)^{n-j}+\frac{1}{(n-1)!}\int_{t_0}^{t}(t-x)^{n-1}f\,[t,u(t)]dx. \qquad (28)$$

Existence and uniqueness of the solution to the Cauchy type problem

In this section, we establish the existence of a unique solution to the Cauchy type problem (5) and (6) in the space $L^{\alpha}(t_0,t_n)$ and an additional Lipschitzian-type condition on $f\,[t,u]$ with respect to the second variable:

$$\forall\, t\in(t_0,t_n],\forall u_1,u_2\in G\subset C,\ \left|f\left[t,u_1\right]-f\left[t,t_2\right]\right|\le A\left|u_1-u_2\right|,\ \ A>0. \qquad (٢٩)$$

where A does not depend on $t\in(t_0,t_n]$.First we derive a unique solution to the Cauchy-problem (20)-(21).

Theorem 2. Let $\alpha>0, n=-\left[\alpha\right]$. Let $G\subset R$ be an open set and $f:\left[t_0,t_n\right]\times G\to R$ be a function such that $\forall u\in G,\ f\left[t,u\right]\in L(a,b)$ and the condition (29) is satisfied.Then there exists a unique solution $u(t)$ to the Cauchy type problem (20)-(21) in the space $L^{\alpha}\left(t_0,t_n\right)$.

Proof. First we prove the existence of a unique solution $u(t)\in L\left(t_0,t_n\right)$. Accordingto Theorem 1, it is sufficient to prove the existence of a unique solution $u(t)\in L\left(a,b\right)$ to the nonlinear Volterra integral equation (11). For this we apply the known method, for nonlinear Volterra integral equations, of proving firstthe result on a part of the interval $\left[t_0,t_n\right]$. Equation (11) makes sense in any interval $\left[t_0,t_1\right]\subset\left[t_0,t_n\right](t_0<t_1<t_n)$. Choose t_1 such that the inequality

$$A\frac{(t_1-t_0)^{\alpha}}{\Gamma(\alpha+1)}<1,$$
(٣٠)

holds, and then prove the existence of a unique solution $u(t)\in L(t_0,t_1)$ to the equation (11) on the interval $[t_0,t_n]$. For this we use the Banach fixed pointtheorem for the space $L(t_0,t_1)$, which is clearly a complete metric space with the distance

$$d(u_1,u_2)=\|u_1-u_2\|_1=\int_{t_0}^{t_1}|u_1(t)-u_2(t)|dt.$$
(٣١)

We rewrite the integral equation (11) in the form $(u-T_{m-1}[u]))(t)=(Tu)(t)$, where

$$(Tu)(t)=u_0(t)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f[v,u(v)]}{(t-v)^{1-\alpha}}dv, \tag{32}$$

with

$$u_0(t)=\sum_{j=1}^{n}\frac{l_j}{\Gamma(\alpha-j+1)}(t-t_0)^{\alpha-j}. \quad (٣٣)$$

We have to prove the following:

(1) if $u(t)\in L(t_0,t_1)$ then $(Tu)(t)\in L(t_0,t_1)$,

and

(2) $\forall u_1,u_2\in L(t_0,t_n)$, the following estimate holds:

$$\|Tu_1-Tu_2\|_1\le w\ \|u_1-u_2\|_1, w=A\frac{(t_1-t_0)^{\alpha}}{\Gamma(\alpha+1)}.$$
(٣۴)

It follows from (33) that $u_0(t)\in L(t_0,t_1)$. Since $f[t,u]\in L(a,b)$, the integral in the right-hand side of (32) also belongs to $L(t_0,t_1)$, and hence $(Tu)(t)\in L(t_0,t_1)$. Now we prove the estimate (34). By (32)-(33) and (17), using the Lipschitzian condition (29) and applying the relation (19) (with $b=t_1$ and $g(t)=f[t,u_1(t)]-f[t,u_2(t)]$, we have

$$\begin{aligned}\|Tu_1-Tu_2\|_{L(t_0,t_n)} &\le \left\|L^{\alpha}_{t_0+}(|f[t,u_1(t)]-f[t,u_2(t)]|)\right\|_{L(t_0,t_n)}\\ &\le A\left\|L^{\alpha}_{t_0+}[|u_1(t)-u_2(t)]|\right\|_{L(t_0,t_n)}\\ &\le A\frac{(t_1-t_0)^{\alpha}}{\Gamma(\alpha+1)}\|u_1(t)-u_2(t)]\|_{L(t_0,t_n)},\end{aligned}$$

which yields (34). In accordance with (30), $0 < w < 1$, and hence there exists a unique solution $u^*(t) \in L(t_0, t_n)$ to the equation (11) on the interval $[t_0, t_1]$. The solution u^* is obtained as a limit of a convergent sequence $(T^k u_0^*)(t)$:

$$\lim_{k \to \infty} \left\| T^k u_0^* - u^* \right\|_{L(t_0, t_1)} = 0,$$

(۳۵)

where $u_0^*(t)$ is any function in $L(t_0, t_n)$. If at least one $l_m \neq 0$ in the initial condition (21), we can take $u_0^*(t) = u_0(t)$ with $u_0(t)$ defined by (33). By (32), the sequence $(T^k u_0^*)(t)$ is defined by the recursion formulas

$$(T^k u_0^*)(t) = u_0^*(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|f[x, (T^k u_0^*)(x)]|}{(t-x)^{1-\alpha}} dx, \quad k = 1, 2, \ldots .$$

If we denote $u_k(t) = (T^k u_0^*)(t)$, then the last relation takes the form

$$(u - T_{m-1}[u])(t)_k(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|f[x, u_{k-1}(x)]|}{(t-x)^{1-\alpha}} dx, \quad m \in N,$$

(36)

and (35) can be rewritten as follows:

$$\lim_{k \to \infty} \left\| (u - T_{m-1}[u])(t)_k - (u - T_{m-1}[u])(t)^* \right\|_{L(t_0, t_n)} = 0. \tag{۳۷}$$

This means that we actually applied the method of successive approximations to find a unique solution $(u - T_{m-1}[u])(t)^*(t)$ to the integral equation (11) on $[t_0, t_1]$. Next we consider the interval $[t_1, t_2]$, where $t_2 = t_1 + h_1$ and $h_1 > 0$ are suchthat $t_2 < t_n$. Rewrite the equation (37) in the form

$$(u - T_{m-1}[u])(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \frac{f[x, u_{k-1}(x)]}{(t-x)^{1-\alpha}} dx + \sum_{j=1}^{n} \frac{l_j}{\Gamma(\alpha - j + 1)} (t - t_0)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \frac{f[x, u(x)]}{(t-x)^{1-\alpha}} dx . \tag{۳۸}$$

Since the function $u(t)$ is uniquely defined on the interval $[t_0, t_1]$, the last integral can be considered as the known function, and we rewrite the last equation as

$$(u - T_{m-1}[u])(t) = u_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \frac{f[x, u(x)]}{(t-x)^{1-\alpha}} dx . \tag{۳۹}$$

where $u_{01}(t)$ as defined by

$$u_{01}(t) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} (t - t_0)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \frac{f[x, u(x)]}{(t-x)^{1-\alpha}} dx , \tag{۴۰}$$

is the known function. Using the same arguments as above, we derive that there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})^{*}\in L(\mathrm{t}_0,\mathrm{t}_n)$ to the equation (11) on the interval $[\mathrm{t}_1,\mathrm{t}_2]$. Taking the next interval $[\mathrm{t}_2,\mathrm{t}_3]$, where $\mathrm{t}_3=t_2+h_2$ and $h_2>0$ are such that $\mathrm{t}_3<t_n$, and repeating this process, we conclude that there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})^{*}\in L(\mathrm{t}_0,\mathrm{t}_n)$ for the equation (11) on the interval $[\mathrm{t}_0,\mathrm{t}_n]$.

Thus, there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})=(u-\mathrm{T}_{m-1}[u])(\mathrm{t})^{*}\in L(\mathrm{t}_0,\mathrm{t}_n)$ to the Volterra integral equation (11) and hence to the Cauchy type problem (20)-(21). To complete the proof of Theorem 2, we must show that such a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$ belongs to the space $L^{\alpha}(\mathrm{t}_0,\mathrm{t}_n)$. In accordance with (17), it is sufficient to prove that $(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$. By the above proof, the solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$ is a limit of the sequence $(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$:

$$\lim_{k\to\infty}\left\|(u-\mathrm{T}_{m-1}[u])(\mathrm{t})_k-(u-\mathrm{T}_{m-1}[u])(\mathrm{t})\right\|_1=0, \tag{41}$$

with the choice of certain $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})_k$ on each $[\mathrm{t}_0,\mathrm{t}_1],\dots,[\mathrm{t}_{L-1},\mathrm{t}_n]$. By (20) and (29) we have

$$\left\|(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})_k-(\mathrm{D}^{\alpha}_{t_0+}u\right\|_1=\left\|f[\mathrm{t},(u-\mathrm{T}_{m-1}[u])(\mathrm{t})_k-\mathrm{f}[\mathrm{t},u]]\right\|_1\le A\left\|u_k-u\right\|_1.$$
(۴۲)

Thus, by (41), we get

$$\lim_{k\to\infty}\left\|(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})_k-(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})\right\|=0, \tag{43}$$

and hence $(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})\in L(\mathrm{t}_0,\mathrm{t}_n)$. This completes the proof of Theorem.

Corollary 2. Let $\mathrm{n}\in N$, let $\mathrm{G}\subset C$ be an open set and let $f:[\mathrm{t}_0,\mathrm{t}_n]\times\mathrm{G}\to C$ be a function such that $\forall(u-\mathrm{T}_{m-1}[u])(\mathrm{t})\in\mathrm{G},\ f[\mathrm{t},u]\in\mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$ and (29) holds. Then there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t})$ to the Cauchy problem (9) in the space $\mathrm{L}^{\alpha}(\mathrm{t}_0,\mathrm{t}_n)$:

$$\mathrm{L}^{n}(\mathrm{t}_0,\mathrm{t}_n)=\{u\in\mathrm{L}(\mathrm{t}_0,\mathrm{t}_n):(u-\mathrm{T}_{m-1}[u]))(\mathrm{t})^{(\mathrm{n})}\in\mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)\}.$$
(۴۴)

Theorem 3. Let $\alpha\in C,\ n-1<R(\alpha)<n\ (n\in N)$. Let G be an open set in C and let be a function such that $f[\mathrm{t},u]\in\mathrm{L}(\mathrm{t}_0,\mathrm{t}_n)$ for any $u\in G$ and the condition (29) holds. Then there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t}))(\mathrm{t})$ to the Cauchy type problem (5)-(6) in the space $\mathrm{L}^{\alpha}(\mathrm{t}_0,\mathrm{t}_n)$ defined in (17). In particular, if $0<R(\alpha)<1$, then there exists a unique solution $(u-\mathrm{T}_{m-1}[u])(\mathrm{t}))(\mathrm{t})$ to the Cauchy type problem (10)such that

$$(\mathrm{D}^{\alpha}_{t_0+}(u-\mathrm{T}_{m-1}[u])(\mathrm{t}))(\mathrm{t})=f[\mathrm{t},u(\mathrm{t})](0<\mathrm{R}(\alpha)<1)(\mathrm{D}^{1-\alpha}_{t_0+}u)(\mathrm{t}_0+)=\mathrm{t}_n\in C, \quad (۴۵)$$

in the space $L^{\alpha}(t_0, t_n)$.

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2, if we use the inequality

$$A\frac{(t_1 - t_0)^{R(\alpha)}}{R(\alpha)\left|\Gamma(\alpha)\right|} < 1.$$

instead of the one in (30).

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