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On the integer programming formulation for k edge-disjoint L-hop-constrained paths (kHPP), for L=2, 3.

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ABSTRACT

The k edge-disjoint L-hop-constrained paths problem consists in finding a minimum cost subgraph such that between two given nodes s and t there exist at least k edge-disjoint paths of at most L edges. We give an integer programming formulation for this problem.

KEYWORDS: Integer programming, Edge-disjoint, Hop-constrained, polyhedron.

1 INTRODUCTION

Given a graph G = (N, E) with s, t ∈ N, a L-st-path in G is a path between s and t of length at most L, where the length of a path is the number of its edges (also called hops). Given a function c: E → ℝ which associates a cost c(e) to each edge e ∈ E, the k edge-disjoint L-hop-constrained paths problem (kHPP) is to find a minimum cost subgraph such that between s and t there exist at least k edge-disjoint L-st-paths. In this paper, we give an integer programming formulation for the (kHPP).

2 DEFINITIONS AND PRELIMINARY INTRODUCTIONS

2.1 Definition

Given a graph G = (N, E) and an edge subset F ⊆ E, the 0 - 1 vector $x^F \in R^E$, such that $x^F(e) = 1$ if e ∈ F and $x^F(e) = 0$ otherwise, is called the incidence vector of F. the convex hull if the incidence vectors of the solutions to the kHPP on G, denoted by $P_k(G)$, will be called the kHPP polytope.

2.2 Definition

Given a vector w ∈ $\mathbb{R}^E$ and an edge subset F ⊆ E, we let w(F) = $\sum_{e \in F} w(e)$. For two node subsets W_1, W_2 ⊂ N, we note [W_1, W_2] the set of edge having one node in W_1 and the other in W_2. If $W_1 = \{w_1\}$, we will write [w_1, W_1] for [$\{w_1\}$, W_1]. If W ⊂ N is a node subset of G, we denote N\W by $\bar{W}$.

2.3 Definition

The set of edge that have only one node in W is called a cut and denoted by δ(W). we will write δ(v) for δ({v}). A cut δ(W) such that $s \in W$ and $t \in \bar{W}$ will be called an st-cut.

If x^F is the incidence vector of the edge set F of a solution to the kHPP, then clearly x^F satisfies the following inequalities:

$x(\delta(W)) \geq k$ for all st-cut δ(W) (1)

$0 \leq x(e) \leq 1$ for all $e \in E$. (2)

Inequalities (1) will be called st-cut inequalities and inequalities (2) trivial inequalities. In [4], Dahl introduces a class of inequalities valid for the dominant of the hop-constrained path problem. For the special case of L= 2, they are as follows.

Let V_0, V_1, V_2, V_3 be a partition of N such that $s \in V_0$, $t \in V_3$ and $V_i \neq \emptyset$ for i=1,2. Let T be the set of edge e= uv where $u \in V_i$, $v \in V_j$ and |i - j| > 1. Then the equality

$x(T) \geq 1$

is valid for the 2-path polyhedron. Using the same partition, this inequality can be generalized in a straightforward way to the kHPP polytope as

$x(T) \geq k$. (3)

The set T is called a 2-path-cut and a constraint of type (3) is called a 2-path-cut inequality. See Fig.1 for an example of a 2-path-cut inequality with V_0={s} and V_3= {t}.

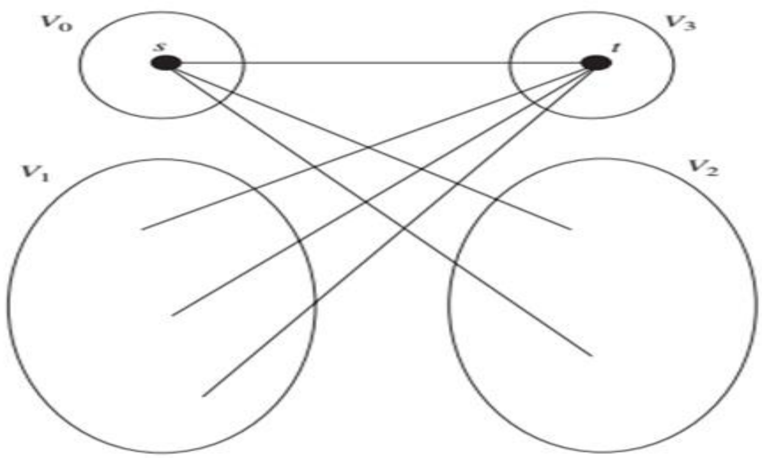

Figure 1. Support graph 2-path-cut inequality.

Note that the 2-path-cut T intersects each 2-st-path in exactly one edge. Let E_1 be the set of edges involved in a 2-st-path in G. Thus, E_1 consists of the edge in [s, t] and [s, v], [v, t] for all those nodes v for which G contains these edges. Let G_1= (N, E_1) be the subgraph of G induced by E_1.

Observe that it is equivalent to consider the kHPP on G and on G_1. More precisely an optimal solution in G will consist of negative costs, if any. Also, it is hard to see that T (in G) corresponds to the st-cut $\delta(V_0 \cup V_1)$ in G_1. Therefore, we will consider the inequalities

$x(\delta_{G1}(W)) \geq k$ for all $W \subset N$, $s \in W$, $t \notin W$ (4)

where $\delta_{G_1}(W)$ stands for the cut induced by W in G_1. Clearly, inequalities (4) dominate inequalities (1) and (3). Let $Q_k(G)$ be a solution set of the system given by inequalities (2), (4). In the next section, we show that inequalities (2), (4), together with inequality constraints, give an integer programming formulation.

3 FORMULATION

In this section, we show that the trivial inequalities and inequalities (4), together with the inequality constraints, suffice to formulate the kHPP as a 0 - 1 linear program. To this aim, we first give a lemma.

3.1 Lemma

Let G = (N, E) be a graph, and s, t two nodes of N, and L ∈ {2, 3}. suppose that there do not exist k edge-disjoint L-st-paths in G, with k ≥ 2. Then there exists a set of at most k - 1 edges that intersects every L-st-path.

Proof. We first show the statement for L= 3. Consider the capacitated directed graph D= (N', A) obtained from G in the following way. The set N' consisits of a copy s', t' of s, t and two copies N_1, N_2 of N\{s, t}. For u ∈ N\{s, t}, let u_1, u_2 be the corresponding nodes in N_1 and N_2, respectively. To each edge e ∈ [s, u], with u ∈ N\{s, t}, we associate an arc e' from s' to u_1 of capacity 1. To each edge e ∈ [v, t], with v ∈ N\{s, t}, we associate an arc e' from v_2 to t' of capacity 1. For an edge e ∈ [u, v], with u, v ∈ N\{s, t}, we consider two arcs, one from u_1 to v_2 and the other from v_1 to u_2, both of capacity1. Finally, we consider in D an arc from s' to t' of capacity1 for every edge in [s, t] and an arc from each node of N_1 to its peer in N_2 with infinite capacity (see Figure 2 for an illustration). Note that multiple edges in G yield multiple arcs in D. Observe that is a one-to-one correspondence between the 3-st-paths in G and the directed $s't'$-paths in D.

Now consider a maximum flow $\phi \in \mathbb{R}_+^E$ from s' to t' in D. As the capacities of D are integer, ϕ can be supposed to be integer. Hence the flow value of each arc of capacity 1 either 0 or 1. We claim that ϕ can be chosen so that no two arcs (u_1, v_2) and (v_1, u_2), corresponding to the same edge uv in G, have a positive value. Indeed, suppose that ϕ (u_1, v_2) =1 and ϕ(v_1, u_2) =1. Let $\phi' \in \mathbb{R}_+^E$ be the flow given by

$$\phi(e) = \begin{cases} \phi(e) + 1 & \textit{if } e \in \{(u_1, u_2), (v_1, v_2)\}, \\ 0 & \textit{if } e \in \{(u_1, v_2), (v_1, u_2)\}, \\ \phi(e) & \textit{otherwise.} \end{cases}$$

As (u_1, u_2) and (v_1, v_2) have infinite capacity and the flow going into u_2 and v_2 has not changed, ϕ' is still feasible. Moreover, ϕ' has the same value as ϕ.

As a consequence, an $s't'$-flow of value q in D corresponds to q edge-disjoint 3-st-paths in G. Since there do not exist, in G, k edge-disjoint 3-st-paths, the maximum flow in D is of value at most k - 1. Hence a minimum st-cut in D is of value at most k-1 as well. Observe that such a cut does not contain arcs with infinite capacity. Hence a minimum cut corresponds to a set of at most k - 1 edges that intersects all the 3-st-paths of G, and the proof for L= 3 is complete.

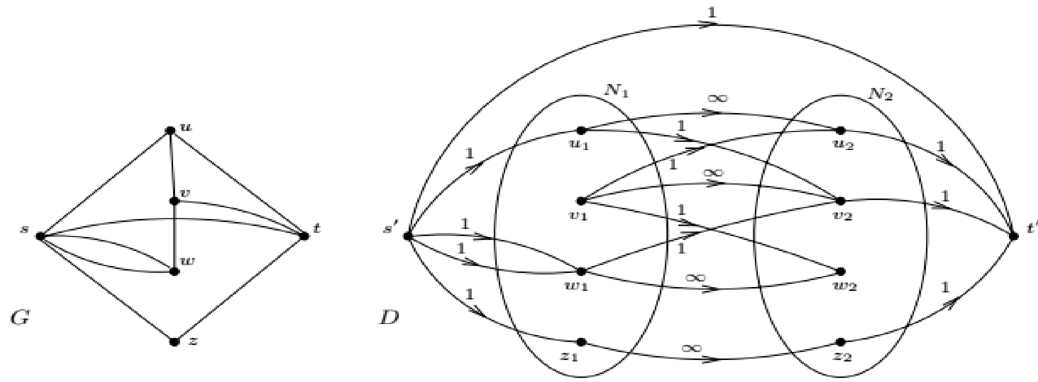

Figure 2.

If L= 2, then we can similarly show the statement by considering the digraph D= (N',A), where N' is a copy of N and to every edge e ∈ [s, u] (resp.,[u, t]), where u ∈ N\{s, t}, corresponds an arc e' from s' to u' (resp., u' to t') of capacity 1 in D. Here u' is the copy of u in N' for every u ∈ N.■

3.2 Theorem

Let G = (N, E) be a graph and k ≥ 2. Then the kHPP is equivalent to the integer program

$$\text{Min}\{cx: x \in Q_k(G), x \in \{0, 1\}^E\}.$$

Proof. To prove the theorem, it is sufficient to show that every 0 – 1 solution x of induces solution of the kHPP. To see the overall proof of this theorem, see [1].

4 SOLVABILITY AND CONCLUDING REMARKS

Note that, if the graph has no parallel edges, the problem can also be solved polynomially by enumerating the (at most) |N| - 1 different 2-st-paths in G and picking the k of these paths with smallest cost. In fact, for this problem, we can suppose that the underlying graph does not contain multiple edges. A natural extension of the kHPP is to consider paths of length at most L where L is a fixed integer. The case studied in this paper corresponds to the case where L=2, 3.

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