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The maximum number of Steiner triple systems with one parallel class in common

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ABSTRACT

Let $D^*(u,k)$ be the maximum number m such that there exist m STS(3u)s $(S,B_1),\dots,(S,B_m)$ such that for each $i \neq j$, $B_i \cap B_j = A$, $|A| = u + k$, where u of the common triples form a parallel class. In this paper, we determine the number $D^*(2n+1,0)$ for each $n \equiv 0,1 \pmod 3$.

KEYWORDS: Quasigroup, Steiner triple system, Parallel class

1 INTRODUCTION

Let X be a set of v points. A (2,3)-packing on X is a pair (X,A), where A is a collection of 3-subsets of X called triples (blocks), such that every pair of distinct elements of X is contained in at most one triple of A. The leave of a (2,3)-packing (X,A) is the graph (X,E) where E consists of all the pairs which do not appear in any block of A. A Steiner triple system (STS) is a (2,3)-packing (S,B) such that its leave is the empty set, i.e. every 2-subset of S is contained in exactly one triple of B. The number |S| is called the order of Steiner triple system. It is well-known that a Steiner triple system of order v exists if and only if $v \equiv 1,3 \pmod 6$. Let (S,B) be a STS(v). A subset P of B is called a parallel class if P partitions S. An STS(v) is called resolvable if all the triples can be partitioned into parallel classes. A resolvable STS(v) is usually called a Kirkman triple system of order v and denoted by KTS(v). The necessary and sufficient condition for the existence for the existence of a KTS(v) is $v \equiv 3 \pmod 6$.

Two STSs (KTSs) (S,B_1) and (S,B_2) are said to intersect in k triples provided $|B_1 \cap B_2| = k$, Two STSs (KTSs) (S,B_1) and (S,B_2) are disjoint if $|B_1 \cap B_2| = 0$. The intersection problem for STSs (KTSs) can be considered in several different types of questions. Two most important of these types of questions are presented in the following.

Question 1: Determine the set Jm(v) (JmR(v)) of all integers k such that there exists a collection of m STS(v)s mutually intersecting in the same set of k triples.

Question 2: Determine the number D(v,k), the maximum number of STS(v)s such that any two of them have exactly k triples in common, these k triples are contained in each of the STSs.

Lindner and Rosa [10] completely determined the set J2(v) and Milici and Quattrocchi [14] determined the set J3(v) for all admissible values of v. The problem of determining the set J2R(v) has been solved by chang and Lo Faro in [2] except for only some undecided cases. Recently, the set J3R(v) has been characterized in [1] except for some values. Y. Li et al in [7] determined J1[u] the set of all integers k such that there is a pair of KTS(3u)s with a common parallel class intersecting in k+u triples, u of them being the triples of the common parallel class. For more studying on the intersection problem for Steiner systems, see [3,6,17].

Milici and Quattrocchi in [15] determined D(v,k) for $k=t_v-m$ with m≤11 and most admissible v. The first author [13] proved that $D(v,t_v-13)=3$ for every admissible v≥15. The problem of determining the value of $D(v,t_v-14)$ has been solved in [16] for all admissible v≥13. The following theorem has been proved in [11,12,19].

Theorem 1. [11,12,19] For v≡1,3 (mod 6), v≠7, D(v,0)=v-2 and D(7,0)=2.

Let D*(u,k) be the maximum number m such that there exist m STS(3u)s $(S,B_1),\ldots,(S,B_m)$ such that for each i≠j, $B_i \cap B_j = A$, |A|=u+k, where u of the common triples form a parallel class. In this paper, we determine the number D*(2n+1,0) for each n≡0,1 (mod 3).

2 PAPER FORMAT

The purpose of this section is to introduce the methods for constructing Steiner triple systems using special structures named quasigroups.

A quasigroup of order n is a pair $(Q,\circ)$, where Q is a set of size n and " $\circ$ " is a binary operation on Q such that for each pair of elements $a,b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. A quasigroup $(Q,\circ)$ is said to be commutative if for each pair of elements $a,b \in Q$, $a \circ b = b \circ a$. It is said to be idempotent if for each $a \in Q$, $a \circ a = a$. Let Q={1,2,…,n} and let F be a 1-factor on the set Q. The two element subsets of F are called holes. A quasigroup with holes F is a quasigroup $(Q,\circ)$ of order 2n in which for each $h \in F$, $(h,\circ)$ is a subquasigroup of $(Q,\circ)$. The following is a quasigroup with holes F ={{1,2},{3,4},{5,6}} of order 6.

∘	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

Theorem 2. [9] For all $n \geq 3$, there exists a commutative quasigroup of order 2n with holes F where F is a 1-factor on the set {1,2,…,2n}.

In the next two constructions, we use commutative quasigroup with holes F.

Construction 1. [9] Let $(\{1,2,\dots,2n\},\circ)$ be a commutative quasigroup of order 2n with holes F. Then $(\{\infty_1,\infty_2,\infty_3\}\cup(\{1,2,\dots,2n\}\times\{1,2,3\}),B)$ is a STS(6n+3), where B is defined by

1. For $\{a,b\}\in F$ let $B_{a,b}$ contain the triples in a STS(9) on the symbols $\{\infty_1,\infty_2,\infty_3\}\cup(\{a,b\}\times\{1,2,3\})$ in which $\{\infty_1,\infty_2,\infty_3\}$ is a triple, and let $B_{a,b}\subseteq B$.
2. For each $1\le a<b\le 2n$ and $\{a,b\}\notin F$ place the triples
$$\{(a,1),(b,1),(a\circ b,2)\},\{(a,2),(b,2),(a\circ b,3)\},\{(a,3),(b,3),(a\circ b,1)\}.$$

Construction 2. Let $(\{1,2,\dots,2n\},\circ)$ be a commutative quasigroup of order 2n with holes F. Then $(\{\infty_1,\infty_2,\infty_3\}\cup(\{1,2,\dots,2n\}\times\{1,2,3\}),B)$ is a STS(6n+3), where B is defined by replacing the triples of type (2) in Construction 1 with the triples

(a) For each $\{a,b\}\notin F$ and $1\le a<b<a\circ b\le 2n$ place the triples
$$\{(a,1),(b,1),(a\circ b,1)\},\{(a,2),(b,2),(a\circ b,2)\},\{(a,3),(b,3),(a\circ b,31)\}$$
(b) For each $\{a,b\}\notin F$ and $1\le a<b\le 2n$ place the triples
$$\{(a,1),(b,2),(a\circ b,3)\},\{(a,2),(b,1),(a\circ b,3)\}$$

Proof. It is easy to check that if S is a set of size v and B is a set of 3-subsets of S such that each pair of distinct elements of S belongs to at least one triple in B and $|B|=\frac{v(v-1)}{6}$, then (S,B) is a Steiner triple system of order v. Let

$$S=\{\infty_1,\infty_2,\infty_3\}\cup(\{1,2,\dots,2n\}\times\{1,2,3\}).$$

We begin proof by counting the number of triples in B. The number of 2-subsets $\{a,b\}\notin F$ is $\binom{2n}{2}-n$. Then the number of triples of type (a) is $3\times\frac{n(2n-2)}{3}$ and type (b) is $2n(2n-2)$. On the other hand, for each 2-subset $\{a,b\}\in F$ there exist the triples of a STS(9) in B. The number of triples in a STS(9) is 12 and $|F|=n$, but the triple $\{\infty_1,\infty_2,\infty_3\}$ is counted in each of n STS(9)s, so

$$|B|=3\times\frac{n(2n-2)}{3}+2n(2n-2)+12n-(n-1).$$

Therefore B contains the right number of triples and so it remains to show that each pair of distinct symbols in S occurs together in at least one triple of B. Let x and y be such a pair of symbols. We consider all of possible cases.

Suppose that $x=\infty_i$ and $y=(a,j)$. Since there exists an element b such that $\{a,b\}\in F$, then x,y belong to a triple of STS(9) $(\{\infty_1\infty_2\infty_3\}\cup(\{a,b\}\times\{1,2,3\}),B_{a,b})$.

Supose that $x=(a,i)$ and $y=(b,j)$. If $\{a,b\}\in F$, then x,y belong to a triple of STS(9)) $(\{\infty_1\infty_2\infty_3\}\cup(\{a,b\}\times\{1,2,3\}),B_{a,b})$. If $\{a,b\}\notin F$, there exist three following cases:

(1) If i=j, then x,y belong to a triple of type (a).
(2) If i=1 and j=2 or i=2 and j=1, then x,y belong to a triple of type (b).
(3) If i=1 and j=3 or i=2 and j=3, for example i=1, since there exists an element c such that $a\circ c=b$, then $x,y\in\{(a,1),(c,2),(b,3)\}$.

Then the assertion follows.

3 The value of $D^*(2n+1,0)$

In this section, we determine the value of $D^*(2n+1,0)$ for each $n \equiv 0,1 \pmod 3$ except n=3.

Theorem 3. For each positive integer number n, $D^*(2n+1,0) = 6n-3$.

Proof. Let S be a set of 6n+3 elements and let $(S,B_1),\ldots,(S,B_t)$ be t Steiner triple systems mutually intersecting in a parallel class named P. Suppose that {x,y,z}and {u,v,w} are two blocks of P. The third element in block containing the pair x,u must be distinct in each of t systems and this element belongs to $S\backslash\{x,y,z,u,v,w\}$. Then $t \le 6n-3$.

Since $n \equiv 0,1 \pmod 3$, then $2n+1 \equiv 1,3 \pmod 6$. Suppose that $S = \{1,2,\ldots,2n+1\}$. By Theorem 2 for $n \ge 4$ there exist 2n-1 disjoint Steiner triple systems of order 2n+1 $(S,B_1),\ldots,(S,B_{2n-1})$ on the set S. Let F_i be the 1-factor on the set {1,2,…,2n} such that for each 2-subset $\{a,b\} \in F_i$, $\{a,b,2n+1\} \in B_i$ for i=1,2,…,2n-1. Suppose that $(Q,\circ_i)$ is the quasigroup of order 2n obtained from Steiner triple system (S,B_i) with holes F_i for i=1,2,…,2n-1.

Let

$$S' = \{\infty_1,\infty_2,\infty_3\} \cup (\{1,2,\ldots,2n\} \times \{1,2,3\})$$

We use a_i instead of (a,i) and abc instead of the block $\{a,b,c\}$. For $1 \le j \le 2n-1$, k=1,2,3 and $\{a,b\} \in F_j$, let $A_{a,b}^{j,k}$ contains the triples in a STS(9) on the symbols $\{\infty_1\infty_2\infty_3\} \cup (\{a,b\} \times \{1,2,3\})$ with the specified triples

$$A_{a,b}^{j,1} = \cup_{i=1}^{3}\{\infty_1 a_i b_i, \infty_2 a_i b_{i+1}, \infty_3 a_i b_{i+2}\} \cup \{\infty_1\infty_2\infty_3, a_1a_2a_3, b_1b_2b_3\},$$

$$A_{a,b}^{j,2} = \cup_{i=1}^{3}\{\infty_1 a_i b_{i+1}, \infty_2 a_i b_{i+2}, \infty_3 a_i b_i\} \cup \{\infty_1\infty_2\infty_3, a_1a_2a_3, b_1b_2b_3\},$$

$$A_{a,b}^{j,3} = \cup_{i=1}^{3}\{\infty_1 a_i b_{i+2}, \infty_2 a_i b_i, \infty_3 a_i b_{i+1}\} \cup \{\infty_1\infty_2\infty_3, a_1a_2a_3, b_1b_2b_3\},$$

And

$$C_{a,b}^{j,1} = \{a_1b_1(a\circ_j b)_2, a_2b_2(a\circ_j b)_3, a_3b_3(a\circ_j b)_1 | 1 \le a < b \le 2n\}$$

$$C_{a,b}^{j,2} = \{a_1b_1(a\circ_j b)_3, a_2b_2(a\circ_j b)_1, a_3b_3(a\circ_j b)_2 | 1 \le a < b \le 2n\}$$

$$C_{a,b}^{j,3} = \left\{a_1b_1(a\circ_j b)_1, a_2b_2(a\circ_j b)_2, a_3b_3(a\circ_j b)_3 \middle| 1 \le a < b < a\circ_j b \le 2n\right\}$$

$$\cup \left\{a_1b_2(a\circ_j b)_3, a_2b_1(a\circ_j b)_3 \middle| 1 \le a < b \le 2n\right\}.$$

Let

$$A_k^j = \cup_{\{a,b\}\in F_j} A_{a,b}^{j,k}, \quad C_k^j = \cup_{\{a,b\}\notin F_j} C_{a,b}^{j,k}.$$

$$B_k^j = A_k^j \cup C_k^j.$$

By Constructions 1 and 2, (S',B_k^j) is a Steiner triple system of order 6n+3 for $1 \le j \le 2n-1$ and k=1,2,3. We claim that for each two Steiner triple systems (S',B_k^j) and (S',B_l^i)

$$B_k^j \cap B_l^i = \{a_1a_2a_3 | 1 \le a \le 2n\} \cup \{\infty_1\infty_2\infty_3\}.$$

That form a parallel class. Obviously $C_k^j \cap C_l^i = \emptyset$ and $A_k^j \cap A_l^i = \{a_1 a_2 a_3 | 1 \leq a \leq 2n\} \cup \{\infty_1 \infty_2 \infty_3\}$ for each $i, j \in \{1,2, \ldots, 2n-1\}$ and $k, l \in \{1,2,3\}$ and $k \neq l$. So

$$B_k^j \cap B_l^i = \{a_1 a_2 a_3 | 1 \leq a \leq 2n\} \cup \{\infty_1 \infty_2 \infty_3\}.$$

Now we show that the claim is true for $1 \leq i < j \leq 2n-1$ and k=l. We prove the claim for k=3 and the other cases are similar. Since $F_i \cap F_j = \emptyset$, then $A_k^j \cap A_l^i = \{a_1 a_2 a_3 | 1 \leq a \leq 2n\} \cup \{\infty_1 \infty_2 \infty_3\}$. We show that $C_k^j \cap C_l^i = \emptyset$. If there exist the pairs $\{a, b\} \notin F_j$ and $\{c, d\} \notin F_i$ such that $a_h b_h (a \circ_j b)_h = c_h d_h (c \circ_i d)_h$, then $\{a, b, a \circ_j b\} = \{c, d, c \circ_i d\}$ and this is in contradiction to our hypothesis. By constructing these 6n-3 Steiner triple systems with one parallel class in common, we conclude that the assertion is true for $n \geq 4$.

For n=1, an STS(9) (S,B) is listed as

$$B = 123,456,789,147,258,369,159,267,348,168,249,357$$

Consider the permutations $\alpha = (123)(456)(789)$ and $\beta = (132)(465)(798)$. It is readily checked that

$B \cap \alpha B = B \cap \beta B = \alpha B \cap \beta B = \{123,456,789\}$.

So $D^*(3,0) = 3$.

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