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Path Cover Number of Rook’s graph

Z. Montazeri, N. Soltankhah

Department of Mathematics
Alzahra University
Vanak Square 19834 Tehran, I.R. Iran
z.montazeri@alzahra.ac.ir; soltan@alzahra.ac.ir

ABSTRACT

The path cover number of graph G, is a set of induced paths of G that cover all vertices of graph G. The path cover number P(G) of graph G is the smallest positive integer k such that there are k vertex-disjoint path occurring as induced subgraphs of G that cover all the vertices of G. There are some relationships between P(G) and some other graph parameters such as minimum rank of graphs and zero forcing number, etc.

Here we investigate the path cover number of Rook's graph and vertex-sum of two Rook's graphs.

KEYWORDS: path covering, path cover number, vertex-sum

1 INTRODUCTION

Let G = (V_G, E_G) be a graph with vertex set V_G and edge set E_G. The order of G is denoted by |G| which is equal to cardinality of V_G. Throughout this paper, all graphs are simple (no loops, no multiple edges), undirected, and have finite nonempty vertex sets. The cartesian product G□H of two graphs G and H has vertex set $V_G \times V_H$ and two vertices (g_1,h_1) and (g_2,h_2) are adjacent if and only if either g_1=g_1 and $h_1h_2 \in E_H$, or h_1=h_1 and $g_1g_2 \in E_G$.

Path covering of a graph G is a family of induced disjoint paths in G that cover all the vertices of the graph. The minimum number of such paths is the path cover number of G and is denoted by P(G). P(G) was first used in the study of minimum rank and maximum eigenvalue multiplicity in [1]. It is also related to some other graph parameters like zero forcing number Z(G) which is the minimum size of a zero forcing set.

Let G be a graph with each vertex given either colour black or white. The colour change rule is: If u is a black vertex and v is the only white neighbor of u, then change colour of v to black and in this case we say u forces v and write u → v. In [3], the authors showed that for any graph the zero forcing number is an upper bound for the path cover number (i.e., for any graph G, P(G) ≤ Z(G)).

2 ROOK'S GRAPH

A graph can be formed from an m × n chessboard by taking the squares as the vertices and two vertices are adjacent if a chess piece situated on one square covered the other. For example Queen's graph, Rook's graph, etc. The Rook's graph R_{mn} is the graph that describes all possible movements of a rook as a chess piece on an empty m × n chessboard. Rooks move either horizontally or vertically. Thus, the Rook's graph can represent the graph $K_n \square K_m$ which is the cartesian product of two complete graphs. A square (i,j) indicates the vertex located on the i-th copy of K_n and the j-th copy of K_m at the same time. All the squares of row i and column j are the neighbors of the square (i,j).

Definition 2.1. [2] A k-colouring of a graph G is a labeling f : V(G) → S, where |S|=k (often we use S=[k]). The labels are colours; the vertices of one colour form a colour class. A k-colouring is proper if adjacent vertices have different labels. A graph is k-colourable if it has a proper k-colouring. The chromatic number χ(G) is the least k such that G is k-colourable.

A Latin rectangle is an m × n matrix (m ≤ n) that has the numbers 1, 2, 3, … , n as its entries with no number occurring more than once in any line (row or column).

Comparing previous definitions gives us the following proposition.

Proposition 2.2. Every m × n latin rectangle represents a proper colouring of R_{mn} which is the graph $K_n \square K_m$.

Proof. It is obvious that every colour class is a (broken) diagonal. So every homochromatic vertices are non-adjacent and the colouring is proper.

Now, we try to introduce a new colouring for Rook's graph such that every colour class is an induced path of the graph.

Consider the graph R_{mn} when m < n and n is even. First place two k for $1 \le k \le \frac{n}{2}$ in the squares (1,2k-1) and (1,2k). Then place the other "k"s in the successive squares in a stairway order along two parallel diagonal which slopes downward and to the right, with the following modifications:

١. When the bottom row for a specific k is reached, stop the diagonal process for this k in this diagonal.

2. When the right-hand column is reached, the next k is put in the left-hand column as if immediately succeeded the right-hand column and continue the diagonal process to reach the bottom row.

Every diagonal has m squares hence every integer appears 2m times in this graph. Each colour class forms an induced path of graph R_{mn}. Every square has exactly one integer so these paths are vertex-disjoint. Therefore we can present following lemma:

Lemma 2.3. For two integers m, n when m < n and n is even, we have:

$$P(R_{mn}) \ge \frac{n}{2}.$$

We illustrate this path covering in figure ١ for graph $K_{10} \square K_5$ (which is the graph $R_{5,10}$).

1	1	2	2	3	3	4	4	5	5
5	1	1	2	2	3	3	4	4	5
5	5	1	1	2	2	3	3	4	4
4	5	5	1	1	2	2	3	3	4
4	4	5	5	1	1	2	2	3	3

Figure 1: The corresponding 5 × 10 chessboard of graph $K_{10}\square K_5$ with the squares of colour k representing the k-th induced path.

We can prove that the equality always holds.

Theorem 2.4. For two integers m, n when m < n and n is even, we have:

$$P(R_{mn}) = \frac{n}{2}.$$

Proof. The upper bound comes from lemma 2.3. It is enough to prove that $P(R_{mn}) > \frac{n}{2}$. Assume, for the sake of contradiction, that $P(R_{mn}) \geq \frac{n}{2}$-1. There are mn squares in m rows. The pigeonhole principle then asserts that there is a path in this covering that has at least three squares in one row. These three squares induced a cycle which is a contradiction. So every induced path in this graph has at most 2m vertices. Then $P(R_{mn}) \geq \frac{n}{2}$ and the equality holds.

The authors guess that for other integer numbers m and n, the following results hold:

Theorem 2.5. For every integer n ≤ 3,

$$P(R_{nn}) = \begin{cases} n & n \text{ is odd} \\ n-1 & n \text{ is even} \end{cases}$$

Theorem 2.6. For two integers m, n when m < n and n is odd, we have:

$$P(R_{mn}) = \left\lfloor \frac{m+n}{2} \right\rfloor.$$

3 VERTEX-SUM OF TWO ROOK'S GRAPHS

Let G and H be two vertex-disjoint graphs and assume that v_1 and v_2 are two vertices of G and H respectively, then the vertex-sum of G and H over v_1 and v_2 is the graph formed by identifying v_1 and v_2 to a unique vertex v, which is denoted by G $+_v$ H.

Theorem 3.1. *[4]* Let G and H be two graphs each with a vertex labeled v. Then the following hold:

1. If there is a minimal path covering of G in which v is a path of length 1 and v is not an end-point in any minimal path covering of H, then

$$P(G +_v H) = P(G) + P(H) - 1.$$

٢. If v is an end-point of some minimal path covering of G but no minimal path covering contains v as a path of length 1 and v is not an end-point in any minimal path covering of H, then

$$P(G +_v H) = P(G) + P(H).$$

٣. If there are minimal path coverings of H and G in which v is an end-point of a path in both, then

$$P(G +_v H) = P(G) + P(H) - 1.$$

٤. If v is neither an end-point of any minimal path covering of H nor the end-point in any minimal path covering of G, then

$$P(G +_v H) = P(G) + P(H) + 1.$$

In vertex-sum of two Rook's graphs, the locations of vertices v_1 and v_2 is not important because we can relabeling the vertices such that v_1 is the square (m,n) in G and v_2 is the square (1,1) in H. Then v_1 and v_2 are end-points in the minimal path covering which we introduced before lemma 2.3. Hence we have the following corollary.

Corollary 3.2. For every integers m, n, r and s ,

$$P(R_{mn} \square R_{rs}) = P(R_{mn}) + P(R_{rs}) - 1.$$

Especially when m < n , r < s and n, s are even, we have:

$$P(R_{mn} \square R_{rs}) = \left\lfloor \frac{n+s}{2} \right\rfloor.$$

Example 3.3. A minimal path covering of graph $(K_8 \square K_5) +_v (K_6 \square K_4)$ is:

1	1	2	2	3	3	4	4					
4	1	1	2	2	3	3	4					
4	4	1	1	2	2	3	3					
3	4	4	1	1	2	2	3					
3	3	4	4	1	1	2	2	2	5	5	6	6
							6	2	2	5	5	6
							6	6	2	2	5	5
							5	6	6	2	2	5

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