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Restricted size Ramsey number of disjoint union of stars versus a complete graph

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Abstract

For given simple graphs F, G and H, we write $F \to (G,H)$ if in every 2-coloring of the edges of F there exists a monochromatic copy of G or H. The Ramsey number $R(G,H)$ is defined as the smallest positive integer n such that $K_n \to (G,H)$. The restricted size Ramsey number $r^*(G,H)$ is defined as the min$\{|E(F)| : F \to (G,H), |V(F)| = R(G,H)\}$. In this note, the exact value of the restricted size Ramsey number of disjoint copies of stars versus a complete graph is determined.

Keywords: Ramsey number, disjoint union of stars, restricted size Ramsey number.

1 Introduction

In this note, we are only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a graph G, we denote its vertex set, edge set, maximum degree, minimum degree and the complement graph of G by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ and $\overline{G}$, respectively. If $v \in V(G)$, we use deg(v) and $N(v)$ to denote the degree and the set of neighborhoods of v in G, respectively. As usual, the star graph on $n+1$ vertices is denoted by $K_{1,n}$ and a complete graph on n vertices is denote by K_n. Also by a *stripe* of size m, mK_2, we mean a graph on $2m$ vertices and m independent edges. A clique in a graph is a subset of vertices such that the induced subgraph on this vertices is a complete graph. In this note, for a given graph G, we use mG to denote the disjoint union of m copies of G. If $Y \subseteq V(G)$, then the induced subgraph of G induced by Y is denoted by $G[Y]$.

For given graphs G, G_1, G_2, we write $G \rightarrow (G_1, G_2)$ if the edges of G are colored in any fashion with colors red and blue, then either the spanning subgraph on edges with color red contains a copy of G_1 or spanning subgraph on edges with color blue contains a copy of G_2. For given graphs G_1, G_2, the *Ramsey number* $R(G_1, G_2)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G_1, G_2)$. The existence of such a positive integer is guaranteed by the Ramsey's classical result [5]. For a survey on Ramsey theory, we refer the reader to the regularly updated survey by Radziszowski [4].

There are several generalizations of Ramsey number and the one that we focus on here is called the restricted size Ramsey number. For given graphs G_1, G_2, the *restricted size Ramsey number* $r^*(G_1, G_2)$ is defined as the $\min\{|E(F)| : F \rightarrow (G_1, G_2), |V(F)| = R(G_1, G_2)\}$. Since the complete graph on $R(G_1, G_2)$ vertices has $\binom{R(G_1,G_2)}{2}$ edges, we obtain trivially that

$$r^*(G_1, G_2) \leq \binom{R(G_1, G_2)}{2}.$$

It is proved that if both G_1 and G_2 are complete graphs, then $r^*(G_1, G_2) = \binom{R(G_1,G_2)}{2}$. The case of complete graph is one of a few cases for which that upper bound is reached. In general, there is very little known about restricted size Ramsey numbers and $r^*(G_1, G_2)$ is known for very little graphs G_1 and G_2. In [2], Faudree and Sheehan determined the exact value of the restricted size Ramsey number of a star and a complete graph.

Theorem 1.1. ([2]) *Let* $\alpha = k(n-1)+1$ *for* $k, n \geq 2$. *Then*

$$r^*(K_{1,k}, K_n) = \begin{cases} \binom{\alpha}{2} - \binom{k}{2} & k \text{ is odd or } k > n, \\ \binom{\alpha}{2} - \frac{1}{2}k(n-1) & \text{Otherwise.} \end{cases}$$

The main aim of this note is to extend the result of Faudree and Sheehan by determining the exact value of the restricted size Ramsey number of disjoint union of stars versus a complete graph. More precisely, we prove the following theorem.

Theorem 1.2. *For positive integers* k, n, t,

$$r^*(tK_{1,k}, K_n) = \begin{cases} \binom{n+2t-2}{2} & \text{if } k = 1, \\ \binom{k(n+t-2)+t}{2} - \binom{k}{2} & \text{if } k > 1 \text{ is odd or } k \geq n+t+\lfloor \frac{t-1}{k} \rfloor, \\ \binom{k(n+t-2)+t}{2} - \frac{k(n-2)}{2} - \lfloor \frac{t(k+1)}{2} \rfloor & \text{otherwise.} \end{cases}$$

2 Proof of the main result

To prove the main result of the paper, we need some lemmas. We begin with the following lemma which was proved in [3].

Lemma 2.1. ([3]) *Let* $k \geq 2$ *and* G *be a graph with* $|E(G)| \geq \binom{k}{2} + 1$, *then either*
(i) G *contains an induced subgraph with* $k+1$ *vertices and minimum degree at least 1 or*
(ii) G *contains a matching* M *with* $|M| = |E(G)|$.

The following result is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let $k \geq 3$ and G be a graph with $|E(G)| \geq \binom{k}{2} + 1$. If k is odd, then G contains an induced subgraph with $k+1$ vertices and minimum degree at least 1.*

Proof. Having applied Lemma 2.1, we may suppose that G contains a matching M with $|M| = |E(G)|$. Let Y be the set of vertices incident to a subset of $\frac{1}{2}(k+1)$ elements of M. Clearly, $G[Y]$ is an appropriate induced subgraph. □

In addition, to prove the main result of the paper, we need the following theorem which determine the exact value of the Ramsey number of a forest versus a complete graph.

Theorem 2.3. ([6]) *Let F be an arbitrary forest and $n(F)$ denote the number of vertices of the largest component of F. Then,*

$$R(F, K_m) = \max_{1 \leq j \leq n(F)} \{(j-1)(m-2) + \sum_{i=j}^{n(F)} ik_i(F)\},$$

where, $k_i(F)$ is number of components of F with exactly i vertices.

As a direct result of the previous theorem, we obtain that if F is a forest which consists of t trees each on k vertices, then $R(F, K_n) = (k-1)(n-2) + kt$.

Now we are ready to prove our main result. First, we compute the exact value of the restricted Ramsey number $r^*(tK_{1,k}, K_n)$ in the case that $k = 1$. Note that in this case $tK_{1,k}$ is isomorphic with a matching of size t, tK_2.

Theorem 2.4. *If $n, t \geq 1$, then*

$$r^*(tK_2, K_n) = \binom{n+2t-2}{2}.$$

Proof. By Theorem 2.3, $R(tK_2, K_n) = n + 2t - 2$ and since $r^*(tK_2, K_n) \leq \binom{R(tK_2,K_n)}{2}$, then

$$r^*(tK_2, K_n) \leq \binom{n+2t-2}{2}.$$

Now, let G be a graph with $|V(G)| = n + 2t - 2$ and $|E(G)| \leq \binom{n+2t-2}{2} - 1$. To see that the restricted size Ramsey number $r^*(tK_2, K_n)$ can not be less than the claimed number, it is sufficient to give a 2-coloring of the edges of G such that $G \nrightarrow (tK_2, K_n)$. As G is a graph on $n+2t-2$ vertices and $|E(G)| \leq \binom{n+2t-2}{2} - 1$, consider an edge $e = uv$ missing from K_{n+2t-2}. Consider a subset $S \subset V(G)$ such that $|S| = 2t - 1$ and $u, v \notin S$. Color all edges contained in S by red and the rest edges by blue. Since $|S| = 2t - 1$ and tK_2 has $2t$ vertices, there is no red copy of tK_2. Since edge e is missing from the blue subgraph of G, the largest clique in $G \setminus S$ is of order $n - 2$ and since subset S is independent in the blue graph, the largest clique in blue graph is of order $n - 1$. Therefore, $G \nrightarrow (tK_2, K_n)$, means that

$$r^*(tK_2, K_n) \geq \binom{n+2t-2}{2},$$

which completes the proof. □

To determine the restricted Ramsey number $r^*(tK_{1,k}, K_n)$ for $k \geq 2$, we need the following lemma.

Lemma 2.5. *Let $k \geq 2$ and H be a graph with $R = R(tK_{1,k}, K_n) = k(n+t-2)+t$ vertices. If*

$$R' = \begin{cases} \binom{k}{2}+1 & \text{if } k \text{ is odd or } k \geq n+t+\lfloor \frac{t-1}{k} \rfloor, \\ \frac{k(n-2)}{2} + \lfloor \frac{t(k+1)}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

and $|E(H)| \geq R'$, then $V(H)$ contains a subset Y with $k+1$ vertices and $\delta(H[Y]) \geq 1$.

Proof. If $k \geq n+t+\lfloor \frac{t-1}{k} \rfloor$, then

$$2(\binom{k}{2}+1) > k(n+t-2)+t$$

and so H does not contains a matching M with $|M| \geq \binom{k}{2}+1$. Therefore when k is odd or $k \geq n+t+\lfloor \frac{t-1}{k} \rfloor$, by Lemma 2.1 and Corollary 2.2, $V(H)$ contains a subset Y with $k+1$ vertices and $\delta(H[Y]) \geq 1$.

Let k even and $k < n+t+\lfloor \frac{t-1}{k} \rfloor$ and assume $V(H)$ does not contains a subset Y with $k+1$ vertices and $\delta(H[Y]) \geq 1$. Since for $k < n+t+\lfloor \frac{t-1}{k} \rfloor$ we have $|E(H)| = \frac{k(n-2)}{2} + \lfloor \frac{t(k+1)}{2} \rfloor + 1 \geq \binom{k}{2}+1$, then by Lemma 2.1 H contains a matching M with $|M| = |E(H)|$. This means that

$$|V(H)| \geq 2|M| \geq 2(\frac{k(n-2)}{2} + \lfloor \frac{t(k+1)}{2} \rfloor + 1) > k(n+t-2)+t = |V(H)|,$$

a contradiction. Therefore, there exists $Y \subseteq V(H)$ such that $|Y| = k+1$ and $\delta(H[Y]) \geq 1$. □

Now, we determine the restricted Ramsey number of disjoint union of stars versus a complete graph.

Theorem 2.6. *For positive integers k, t and $n \geq 2$,*

$$r^*(tK_{1,k}, K_n) = \begin{cases} \binom{n+2t-2}{2} & \text{if } k = 1, \\ \binom{k(n+t-2)+t}{2} - \binom{k}{2} & \text{if } k > 1 \text{ is odd or } k \geq n+t+\lfloor \frac{t-1}{k} \rfloor, \\ \binom{k(n+t-2)+t}{2} - \frac{k(n-2)}{2} - \lfloor \frac{t(k+1)}{2} \rfloor & \text{otherwise.} \end{cases}$$

Proof. The case $k = 1$, follows from Theorem 2.4. Thus, let $k \geq 2$. We consider the following cases.

Case 1: k is odd or $k \geq n+t+\lfloor \frac{t-1}{k} \rfloor$.

Assume that $p = k(n+t-2)+t$ and $G = K_{p-k} + \overline{K}_k$. Clearly, G has $k(n+t-2)+t$ vertices and $\binom{k(n+t-2)+t}{2} - \binom{k}{2}$ edges. We prove that $G \to (tK_{1,k}, K_n)$.

The proof is by induction on t. If $t = 1$, then $G = K_{k(n-1)+1} - K_k$. It is easy to see that $|E(G)| = \binom{k(n-1)+1}{2} - \binom{k}{2}$ and so by Theorem 1.1, $G \to (K_{1,k}, K_n)$. Now, let $t \geq 2$ and consider an arbitrary 2-edge coloring red/blue of G. First, suppose that there is a monochromatic star $K_{1,k}$ whose edges are colored red. Let G' be the graph obtained from G by deleting the vertices of this red star $K_{1,k}$. Now, G' has $k(n+t-3)+t-1$ vertices, which by the induction hypothesis, $G' \to ((t-1)K_{1,k}, K_n)$. If G'

contains a red copy of $(t-1)K_{1,k}$, with our deleted star became a red monochromatic $tK_{1,k}$. Otherwise, G' has a blue K_n, and so $G \to (tK_{1,k}, K_n)$.

Now, assume that there is no red $K_{1,k}$ in G. Delete $k+1$ vertices arbitrary from G and denote the remaining graph by G'. Again, G' has $k(n+t-3)+t-1$ vertices, which by the induction hypothesis, $G' \to ((t-1)K_{1,k}, K_n)$. Since $t \geq 2$ and there is no red $K_{1,k}$ in G, then there exist a blue copy of K_n in G' and so in G. Therefore, $G \to (tK_{1,k}, K_n)$. This observation shows that

$$r^*(tK_{1,k}, K_n) \leq \binom{k(n+t-2)+t}{2} - \binom{k}{2}.$$

Now, to see that the restricted size Ramsey number $r^*(tK_{1,k}, K_n)$ can not be less than the claimed number, let H be a graph on $k(n+t-2)+t$ vertices and $|E(H| \leq \binom{k(n+t-2)+t}{2} - \binom{k}{2} - 1$. We show that $H \nrightarrow (tK_{1,k}, K_n)$.

First, consider the complement graph of H, means $\overline{H}$, which has the same vertex set as H and $|E(\overline{H})| \geq \binom{k}{2} + 1$. By Lemma 2.5, there is a set $Y \subseteq V(\overline{H})$ so that $|Y| = k+1$ and $\delta(\overline{H}[Y]) \geq 1$. this means that the maximum degree in $H[Y]$ is at most $k-1$ and so there is no $K_{1,k}$ in $H[Y]$. Partition the vertices of H into $n-1$ parts $V_1, V_2, \ldots, V_{n-1}$ in the following way. Choose a set $Y \subseteq V(H)$ with $\Delta(H[Y]) \leq k-1$ and set $V_1 = Y$. Let $V_2 \subseteq V(H) \setminus Y$ be a set of size $t(k+1)-1$ and partition the remaining vertices into $V_3, \ldots, V_{n-1}$ with $|V_3| = \cdots, |V_{n-1}| = k$. Color all edges with both ends in V_i, $1 \leq i \leq n-1$, by red and the rest by blue. Clearly the maximum number of red copies of $K_{1,k}$ is $t-1$ and the maximum clique in the blue graph is $n-1$. Therefore, there is no red $tK_{1,k}$ and no blue K_n means that $H \nrightarrow (tK_{1,k}, K_n)$.

Case 2: k is even and $k < n + t + \lfloor \frac{t-1}{k} \rfloor$.

Let $G = K_{k(n+t-2)+t} - F$, where F is a graph isomorphic to a matching of size $\frac{k(n-2)}{2} + \lfloor \frac{t(k+1)}{2} \rfloor$. Clearly, G has $k(n+t-2)+t$ vertices and $\binom{k(n+t-2)+t}{2} - \frac{k(n-2)}{2} - \lfloor \frac{t(k+1)}{2} \rfloor$ edges. We prove that $G \to (tK_{1,k}, K_n)$.

The proof is by induction on t. If $t = 1$, then $G = K_{k(n-1)+1} - (\frac{k(n-1)}{2})K_2$. It is easy to see that $|E(G)| = \binom{k(n-1)+1}{2} - \frac{k(n-1)}{2}$ and so by Theorem 1.1, $G \to (K_{1,k}, K_n)$. Now, let $t \geq 2$ and consider an arbitrary 2-edge coloring red/blue of G. First, suppose that there is a monochromatic star $K_{1,k}$ whose edges are colored red. Let G' be the graph obtained from G by deleting the vertices of this red star $K_{1,k}$. Now, G' has $k(n+t-3)+t-1$ vertices, which by the induction hypothesis, $G' \to ((t-1)K_{1,k}, K_n)$. If G' contains a red copy of $(t-1)K_{1,k}$, with our deleted star became a red monochromatic $tK_{1,k}$. Otherwise, G' has a blue K_n, and so $G \to (tK_{1,k}, K_n)$.

Now, assume that there is no red $K_{1,k}$ in G. Delete $k+1$ vertices arbitrary from G and denote the remaining graph by G'. Again, G' has $k(n+t-3)+t-1$ vertices, which by the induction hypothesis, $G' \to ((t-1)K_{1,k}, K_n)$. Since $t \geq 2$ and there is no red $K_{1,k}$ in G, then there exist a blue copy of K_n in G' and so in G. Therefore, $G \to (tK_{1,k}, K_n)$. This observation shows that

$$r^*(tK_{1,k}, K_n) \leq \binom{k(n+t-2)+t}{2} - \frac{k(n-2)}{2} - \lfloor \frac{t(k+1)}{2} \rfloor.$$

Now, to see that the restricted size Ramsey number $r^*(tK_{1,k}, K_n)$ can not be less than the claimed number, let H be a graph on $k(n+t-2)+t$ vertices and $|E(H| \leq \binom{k(n+t-2)+t}{2} - \frac{k(n-2)}{2} - \lfloor \frac{t(k+1)}{2} \rfloor - 1$. We show that $H \nrightarrow (tK_{1,k}, K_n)$.

First, consider the complement graph of H, means $\overline{H}$, which has the same vertex set as H and $|E(\overline{H})| \geq \binom{k}{2} + 1$. By Lemma 2.5, there is a set $Y \subseteq V(\overline{H})$ so that $|Y| = k + 1$ and $\delta(\overline{H}[Y]) \geq 1$. this means that the maximum degree in $H[Y]$ is at most $k - 1$ and so there is no $K_{1,k}$ in $H[Y]$. Now, partition the vertices of H into sets $V_1, V_2, \ldots, V_{n-1}$ as described in case 1. Color all edges with both ends in V_i, $1 \leq i \leq n - 1$, by red and the rest by blue. Clearly the maximum number of red copies of $K_{1,k}$ is $t - 1$ and the maximum clique in the blue graph is $n - 1$. Therefore, there is no red $tK_{1,k}$ and no blue K_n means that $H \nrightarrow (tK_{1,k}, K_n)$. This observation completes the proof. □

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory With Applications, American Elsevier Publishing Co. INC, (1976).

[2] R. J. Faudree, J. Sheehan, Size Ramsey numbers involving stars, Discrete Math. 46 (1983), 151–157.

[3] M. Miralaeia, G. R. Omidi, M. Shahsiahb, Size Ramsey Numbers of Stars versus Cliques, arXiv:1601.06599v2.

[4] S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* (2014), DS1.

[5] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* 30 (1930), 264-286.

[6] S. Stahl, On the Ramsey number $R(F, K_m)$ where F is a forest, Can. J. Math. (1975), 585-589.