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On Biclique Cover and Partition of the Kneser Graph

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ABSTRACT

Let $bc_d(G)$ (resp. $bp_d(G)$) denote the minimum number of bicliques of G such that every edge of G belongs to at least (resp. exactly) d of these bicliques. Hajiabolhassan and Moazami (2012a) showed existence of a secure frame proof code results from the existence of biclique cover of Kneser graph (KG(t, r)(and vice versa. Also, they (Hajiabolhassan and Moazami, 2012a) showed that using d-biclique cover of Kneser graphs, we can obtain appropriate lower and upper bound for the minimum number of points in cover free family. As was shown by Orlin (1977), determining the exact value biclique covering number is NP-hard. Hence, it is a challenging and interesting problem to determine the exact value of $bc_d(KG(t, r))$ ($bp_d(KG(t, r))$). In this paper, we determine the exact value of $bc_d(KG(t, r))$ ($bp_d(KG(t, r))$) for every r, t, where $2r \leq t$ and some d.

KEYWORDS: Kneser graph, Biclique cover, Biclique partition

1 INTRODUCTION

The Kneser graph KG(t, r) is a graph whose vertices are all r-subsets of a telement set, where two r-subsets are adjacent if and only if their intersection is empty. A biclique of G is a complete bipartite subgraph of G. The d-biclique covering (resp. partition) number $bc_d(G)$ (resp. $bp_d(G)$) of a graph G is the minimum number of bicliques of G such that every edge of G belongs to at least (resp. exactly) d of these bicliques. A binary code Γ of length v and size t is called r-secure frameproof, if for any v-word that is produced by two subsets C_1 and C_2 of Γ of size at most r, then the intersection of these sets is non-empty. Secure frameproof codes are used to protect digital data, computer software, etc (Stinson et al., 2000). Hajiabolhassan and Moazami (2012a) showed existence of a secure frame proof code results from the existence of biclique cover of Kneser graph and vice versa. An (r, w; d)-cover free family (CFF) is a family of subsets of a finite set such that the intersection of any r members of the family contains at least d elements that are not in the union of any other w members. The minimum number of elements for which there exists an (r, w; d)-CFF with t blocks is denoted by N ((r, w; d), t). Also, Hajiabolhassan and Moazami (2012a) showed that using d-biclique cover of Kneser graphs, we can obtain appropriate lower and upper bound for N ((r, w; d), t). For another application and discussion of biclique covering, (see, for example, [(Azadimotlagh and Moazami), (Fleischner et al., 2009), (Haemers 2001), (Hajiabolhassan and Moazami, 2012), (Stinson et al., 2000), (Tuza, 1984)]. As was shown by Orlin (1977), determining the exact value d-biclique covering (partition) number for a graph is NP-hard. Thus the problem of determining the exactvalue of the d-biclique covering (partition) number even for special graphs is a challenging and interesting problem.

2 BICLIQUE COVERING

In this section, we restrict our attention to determine the exact value of d-biclique covering number and d-partition covering number of a graph $KG(t,r)$ for every value of r and t, where $2r \leq t$, and some special value of d.

THEOREM 1. Let r and t be positive integers, where t ≥ 2r. Also, assume that the function $\binom{x}{r}\binom{t-x}{r}$ is maximized for x = t. If $d = \binom{t-2r}{t'-r}$, then

$$bc_d(\mathrm{KG(t,r)}) = bp_d(\mathrm{KG(t,r)}) = \frac{\binom{t}{t'}}{2}$$

PROOF. Set $t'' = \frac{\binom{t}{t'}}{2}$. First, we show that $KG(t,r)$ can be covered by t'' bicliques such that every edge of $KG(t,r)$ is covered by exactly d bicliques. Denote the vertex set of $KG(t,r)$ by set $\binom{[t]}{t'}$. Suppose that A_j is a t'-subset of [t] and ${A_j}^c$ is the complement of the set A_j in $[t]$. Denote the number of these pairs by t''. Now, for every t'-subset A_j of $[t]$, where $1 \ll j \ll t''$, construct the biclique G_j with the vertex set (X_j, Y_j) , where $X_j = \binom{A_j}{r}$ and $Y_j = \binom{{A_j}^c}{r}$. Let UV be an arbitrary edge of $KG(t,r)$, where $|U| = |V| = r$.

In view of the definition of G_j, UV is covered by every G_j with vertex set (X_j, Y_j) , where U is a vertex of X_j and V is a vertex of Y_j. Thus every edge of $KG(t,r)$ is covered by at least d bicliques. One can see that

$$\sum_{j=1}^{t''} |E(G_j)| = \frac{\binom{t}{t'}\binom{t}{r}\binom{t-t'}{r}}{2} \qquad (1)$$

And

$$|E(KG(t,r))| = \frac{\binom{t}{r}\binom{t-r}{r}}{2}. \qquad (2)$$

Now, it is simple to check that

$$\sum_{j=1}^{t''} |E(G_j)| = d|E(KG(t,r))|. \qquad (3)$$

Thus every edge of $KG(t,r)$ is covered by exactly d bicliques. Note that we have actually proved that

$$bp_d(KG(t,r)) \leq t''. \qquad (4)$$

Conversely, one can see that if $l = bc_d(KG(t,r))$, $\{H_1, \ldots, H_l\}$ is an arbitrary optimal $d-$biclique covering of $KG(t,r)$ and $B(KG(t,r))$ is the maximum number of edges among the bicliques of $B(KG(t,r))$, then

$$lB(KG(t,r)) \geq \sum_{j=1}^{l} |E(H_j)| \geq d|E(KG(t,r))|. \qquad (5)$$

So

$$bc_d(KG(t,r)) \geq \frac{d|E(KG(t,r))|}{B(KG(t,r))}. \qquad (6)$$

Also, it is easy to see that

$$bp_d(KG(t,r)) \geq bc_d(KG(t,r)). \qquad (7)$$

Thus

$$bp_d(KG(t,r)) \geq bc_d(KG(t,r)) \geq \frac{d|E(KG(t,r))|}{B(KG(t,r))}. \qquad (8)$$

Also, one can check that the maximum number of edges among the bicliques of the Kneser graph $KG(t,r)$ is

$$\max_{r\leq x\leq t-r}\left[\binom{x}{r}\binom{t-x}{r}\right].$$

Now, in view of the definition of t', we have

$$B\big(KG(t,r)\big)=\binom{t'}{r}\binom{t-t'}{r}. \tag{9}$$

So we have

$$\frac{d\left|E\big(KG(t,r)\big)\right|}{B\big(KG(t,r)\big)}=\frac{\binom{t-2r}{t'-r}\binom{t}{r}\binom{t-r}{r}}{2\binom{t'}{r}\binom{t-t'}{r}}=\binom{t}{t'}=t''. \tag{10}$$

Hence,

$$bp_d(KG(t,r))\geq bc_d\big(KG(t,r)\big)\geq t''. \tag{11}$$

From (4) and (11), we conclude

$$bp_d(KG(t,r))=bc_d\big(KG(t,r)\big)=t''.$$

3 CONCLUSION

In this paper, we determine the exact value of d-biclique covering number and d-partition covering number of a graph $KG(t,r)$ for every value of r and t, where $2r \leq t$, and some special value of d.

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