3TH INTERNATIONAL CONFERENCE ON COMBINATORICS, CRYPTOGRAPHY AND COMPUTATION

Iran University of Science &Technology

An Extended Version of Adomian Decomposition Method for solving ODEs in the Presence of Interval Uncertainties

Mohsen Shahrezaee
Department of Mathematics, Faculty of Science,
University of Imam Hussein (AS), Tehran, Iran
mshahrezaee@iust.ac.ir

Navid Razmjooy
Department of Electrical and Control Engineering,
Tafresh University, Tafresh, Iran
navid.razmjooy@hotmail.com

ABSTRACT

A newly improved extension of the Adomian decomposition method is presented for ODE problems with interval uncertainties. Interval analysis is employed for analyzing the uncertainties in ODE systems including uncertain-but-bounded parameters with lower and upper bounds of uncertainties. Simulation results are applied to a case study and the results are compared with interval Euler and Taylor methods. Final results show that the proposed methodology has a good accuracy to find the proper interval and to effectively handle the wrapping effect to sharpen the range of non-monotonic interval.

KEYWORDS: Interval Analysis; Adomian decomposition method; ODEs; interval uncertainty.

1 INTRODUCTION

In most cases, parameters in the mathematical models of practical problems are considered deterministic; although, there are always some uncertainties in the model parameters [1]. Uncertainty in the models can make an inaccurate or even wrong representation for the model. Uncertainties can be made by different reasons like measurement error, inhomogeneity of the process, round off error, truncation error and etc [2].

This problem leads researches to analyze the problem from a different point of view. There have been different ways that introduced to consider these uncertainties. Among the different methods for analyzing the uncertainties, probabilistic method, fuzzy method, and interval method result in the best.

Interval method among, because of its feature, has a higher class rather than the other methods. In the interval method, uncertainties stand throughout a definite lower and upper bounds; in other words, although the uncertainties quantity is unknown, an interval can be defined to all of them [3].

Ordinary differential equations (ODEs) have a wide range of applications in different fields of science and technology like the systems modeling, optimal control, etc. There have been different techniques which are presented to solve ODEs [6].

Recently, decomposition methods have been utilized as effective, easy, and accurate methods for solving a wide range of problems including linear and nonlinear, partial, deterministic or stochastic differential equations . These methods have a high quality in convergence to achieve accurate solutions [7].

Adomian decomposition method is one of the most popular decomposition methods which can be employed for solving functional differential equations like ordinary differential equations, differential-algebraic equations, non-linear fractional differential equations, delay differential equations and etc.

Adomian decomposition method is always as a decomposition method for solving deterministic problems. In this paper, an improved version of the Adomian method is proposed for solving the ODES in the presence of uncertainties. The main purpose is to keep the final result in a guaranteed interval stable bound even if the parameters in the system are changed in the considered interval uncertainty.

the rest of the paper is organized as follows: in section2, a brief explanation about interval analysis is introduced. In section3, Adomian decomposition method and its approach are presented. Section 4 explains the extended version of the Adomian method which is the original novelty of the paper. Section 5 illustrates the system performance based on a complicated nonlinear ODE and comparing it by popular interval methods and finally, the paper is concluded in section 6.

2 INTERVAL ANALYSIS

Interval analysis provides a set of techniques to keep track interval uncertainties which happen during the computations. The interval set for an interval integer can be presented as follows:

$$X = [\underline{x}, \overline{x}] = \{x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}, \underline{x} \le x \le \overline{x}\}, \quad (1)$$
$$I(\mathbb{R}) = \{X \mid X = [\underline{x}, \overline{x}], \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \le \overline{x}\},$$

where, X is an interval integer over $I(\mathbb{R})$ and $\underline{x}$ and $\overline{x}$ present the lower and upper bounds respectively. The midpoint value, the width of interval number and the radius of an interval integer can be also considered as follows [4]:

$$x_c = \frac{1}{2}(\overline{x} + \underline{x}), x_r = \frac{x_w}{2}, \quad (2)$$

$$x_w = \overline{x} - \underline{x} \quad (3)$$

The basic interval arithmetic operations are described so that the interval guarantees the reliability of interval results. The basic interval analysis operations between two interval integers are given in below:

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \quad (4)$$

$$X - Y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] \quad (5)$$

$$X \times Y = [\min\{\underline{xy}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{xy}\}, \max\{\underline{xy}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{xy}\}] \quad (6)$$

$$X / Y = X \times \frac{1}{Y}, \quad (7)$$
$$\frac{1}{Y} = [\frac{1}{\overline{y}}, \frac{1}{\underline{y}}], 0 \notin [\underline{y}, \overline{y}]$$

$$X^n = \begin{cases} \left[0, \max(\underline{x}^n, \overline{x}^n)\right], & n = 2k \quad, 0 \in [x] \\ \left[\min(\underline{x}^n, \overline{x}^n), \max(\underline{x}^n, \overline{x}^n)\right], & n = 2k, 0 \notin [x] \\ [\underline{x}^n, \overline{x}^n], & n = 2k + 1 \end{cases} \quad (8)$$

The interval function *F* is an inclusion function of *f*, fi $\forall X \in I(\mathbb{R})$, $f(F) \subset F(X)$.

The main purpose of this paper is to find an interval function F from f to achieve an interval form to the Adomian decomposition method.

3 ADOMIAN DECOMPOSITION METHOD

The main idea behind the Adomian method is to decompose the unknown function (i.e. $y(x)$) into an infinite series $y(x)=\sum_{i=0}^{\infty} y_i(x)$ such that $y_0, y_1, \cdots$ are evaluated recursively [7]. In this case, if there is nonlinearity ($N(y(\mathrm{x}))$) in the function, it must be decomposed by the following formula:

$$N(y(x))=\sum_{n=0}^{\infty} A_n \tag{9}$$

where, $A_n = A_n(y_0(x), y_1(x), \ldots, y_n(x))$ are the Adomian polynomials:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)\Big|_{\lambda=0} \qquad n=0,1,2,\ldots \tag{10}$$

Now, consider an ODE as follows:

$$Ly+Ry+Ny=g(x), \tag{11}$$

where, *N* is the nonlinear operator, *L* presents the highest invertible derivative, *R* defines the linear differential operator with order less than *L* and *g* describes the source term. By applying the inverse term " L^{-1} " to the $Ly = g - Ry - Ny$,

$$y=\gamma+f-L^{-1}(Ry)-L^{-1}(Ny), \tag{12}$$

where, γ describes the given conditions and f is the integration of the source term. By considering the eq. (12), the recurrence relation of *y* can be considered as follows:

$$\begin{cases} y_0=\gamma+f, \\ y_1=-L^{-1}(Ry_0)-L^{-1}(Ny_0) \\ \vdots \\ y_{k+1}=-L^{-1}(Ry_k)-L^{-1}(Ny_k), \qquad \mathrm{k}\geq 0 \end{cases} \tag{13}$$

If the series converges to the considered solution, then

$$y=\lim_{M\to\infty}\tilde{y}_M(x), \qquad \tilde{y}_M(x)=\sum_{i=0}^{M} y_i(x). \tag{14}$$

More details about Adomian decomposition and its convergence can be found in [5].

4 INTERVAL ADOMIAN METHOD FOR PROBLEMS WITH UNCERTAINTIES

Consider the following ODE with interval uncertainties in its dynamic and initial conditions:

$$F\left(X, Y', \ldots, Y^{(n)}, \Delta\right)=0, \qquad Y(x_0)=Y_0, \ldots, Y^{(n)}(x_0)=Y_n, \tag{15}$$

where $\Delta = \left(\delta_1, \delta_1, \delta_2, \delta_3, \ldots, \delta_m\right)^T$ describes uncertain parameters and $\delta_k \in [\underline{\delta}_k, \overline{\delta}_k], k=0,1,\ldots,(n+1)$.

The main idea in this paper is to extend the Adomian decomposition method for solving the ODE problems with interval uncertainties.

Consider a standard ODE form as follows:

$$\underbrace{\delta_n Y^{(n)} + \delta_3 Y'' + \ldots + \delta_2 Y' + \delta_1 Y}_{R\{y(x)\}} + \alpha N\{Y(X)\} = \beta G(X), \qquad \alpha\in[\underline{\alpha},\overline{\alpha}],\ \ \beta\in[\underline{\beta},\overline{\beta}], \tag{16}$$

At first, the system should be converted into the uniform mode,

$$\underbrace{\delta_n Y^{(n)} + \delta_3 Y'' + \ldots + \delta_2 Y' + \delta_1 Y}_{R\{y(x)\}} + \tilde{\alpha} N\{Y(X)\} = \tilde{\beta} G(X), \tag{17}$$

where, the term (~) illustrates the division of the coefficients with δ_n. For instance, $\tilde{\beta}$ is $[\beta]/[\delta_n] = [\beta] \times \left[\frac{1}{\overline{\delta}_n}, \frac{1}{\underline{\delta}_n}\right]$.

δ_n is assumed non-singular, *i.e.* $0 \notin \delta_n$. By using the conditions, the system is achieved as follows:

$$Y = P_n + F - L^{-1}(RY) - L^{-1}(Ny), \quad (18)$$

$$P_n = [\underline{p}_n, \overline{p}_n].$$

Where, P_n is an interval polynomial which has been deniatbo from the initial conditions $P_n = Y_0 + Y_1 x + \ldots + Y_n x^n$.

If $P_n = [p_n, p_n]$, then it will be called degenerative. By applying the inverse operator to the source term, *F*,

$$F = [L^{-1}(\beta g(x))] = [\underline{\beta}, \overline{\beta}] \times [L^{-1}(\underline{g}(x)), L^{-1}(\overline{g}(x))], \quad (19)$$

Since by assuming $g = g_c + \delta_g$, g_c and δ_g are the certain and uncertain terms of *g*,

$$\left[L^{-1} \;\; \underline{g}(x) \;\; , L^{-1} \;\; \overline{g}(x) \;\; \right] = \left[L^{-1} \;\; g_c(x) + \left[\delta_g\right] \;\; \right] \quad (20)$$

We can use a similar operation for obtaining the linear and nonlinear differential operators. Therefore, the final representation for the extended Adomian method is:

$$\begin{cases} Y_0 = [\gamma] + [\underline{\beta}, \overline{\beta}] \times [L^{-1}(\underline{g}(x)), L^{-1}(\overline{g}(x))], \\ Y_{k+1} = [-L^{-1}(R\underline{y}_k) - L^{-1}(N\underline{y}_k), -L^{-1}(R\overline{y}_k) - L^{-1}(N\overline{y}_k)], \quad \text{k} \geq 0 \end{cases} \quad (21)$$

5 EXPERIMENTAL RESULTS

In the following, to analyze the proposed method's efficiency, a case study is studied and the results are compared with interval Euler and Taylor methods. The algorithms are performed by Matlab 2017a. Consider the nonlinear ODE in the interval $0 \leq x \leq 1$

$$Y''(x) - [\delta_1] x Y^2(x) Y'(x) - [\delta_2] Y^3(x) = e^x - 2e^{3x} - 3xe^{3x},$$

$$Y(0) = 1, \;\; Y'(0) = 1, \;\; 1 \leq \delta_1, \delta_2 \leq 2,$$

Where, $L = \frac{d^2}{dx^2}$, $\breve{R} = 0$, $N_1 Y = Y^2(x) Y'(x)$, $N_2 Y = Y^3(x)$, $g(x) = e^x - 2e^{3x} - 3xe^{3x}$.

Source term in here is approximated by the 4th order Chebyshev approximation as follows,

$$g(x) \approx -32.612\, x^4 - 45.0884\, x^3 - 14.2512\, x^2 - 1.7918\, x + 7.1530$$

Since,

$$Y(x) = 1 + x + [\delta_1] \int_0^x \int_0^x x\, N_1(x)\, dx\, dx + [\delta_2] \int_0^x \int_0^x N_2(x)\, dx\, dx + \int_0^x \int_0^x g(x)\, dx\, dx$$

The Adomian polynomials for decomposition of nonlinear parts, $N_1(x)$ and $N_2(x)$ are:

Table 1. Decomposition of Nonlinear terms using Adomian method

$N_1(x)$	$N_2(x)$
$\text{A}_0(x) = Y_0^2 Y_0'$	$\text{A}_0(x) = Y_0^3$
$\text{A}_1(x) = Y_0^2 Y_1' + 2Y_0 Y_1 Y_0'$	$\text{A}_1(x) = 3Y_0^2 Y_1$
$\text{A}_2(x) = Y_0^2 Y_2' + 2Y_0 Y_1 Y_1' + 2Y_0 Y_2 Y_0'$	$\text{A}_2(x) = 3Y_0^2 Y_2 + 3Y_0 Y_1^2$
$\vdots$	$\vdots$

Therefore, by combining the information above,

$$Y_0(x) = 1 + x + 3.5765\,x^2 - 0.2986\,x^3 - 1.1876\,x^4 - 2.2844\,x^5 - 1.0871\,x^6$$

$$Y_1(x) = [\delta_1]\int_0^x\int_0^x Y_0^{\,2}(x)Y_0'(x)\,dx\,dx + [\delta_2]\int_0^x\int_0^x Y_0^{\,3}(x)\,dx\,dx = [x^2 + 2\,x^3 + 2.8\,x^4 + \ldots$$
$$,2x^2 + 4\,x^3 + \ldots]$$
$$\vdots$$
$$y_n(x) = [\delta_1]\int_0^x\int_0^x Y_{n-1}^{\;\;2}(x)\,Y'_{n-1}(x)\,dx\,dx + [\delta_2]\int_0^x\int_0^x Y_{n-1}^{\;\;3}(x)\,dx\,dx$$

Table 2. Final results of the proposed method and its comparison with other interval based methods and a limited random inputin the considered interval.

Time	*Interval Adomian Method*	*Interval Euler's Method* [13]	*Interval Taylor Method* [14]	*Random value*
0	[1, 1]	[1, 1]	[1, 1]	1
0.4	[1.398, 1.401]	[1.06, 1.37]	[0.99, 1.33]	1.399
0.8	[1.780, 1.819]	[0.27, 0.95]	[-0.67, 0.04]	1.781
1.2	[2.092, 2.307]	Divergent	Divergent	2.092
1.6	[2.234, 2.961]	Divergent	Divergent	2.235
2	[2.042, 3.941]	Divergent	Divergent	2.043

Table 2 shows that interval Euler and Taylor methods fail the interval from "Time=0.8", but the proposed method is total including the random value.

6 CONCLUSION

In this paper, a new methodology for solving ODEs with interval uncertainties is presented. The proposed method is designed based on the Adomian decomposition method which has wide applications for solving large varieties of the ODEs and provides a robust approximation of the solution. For solving the ODEs with uncertainty by Adomian method, interval analysis is employed and utilized to generate an extension version of the Adomian decomposition method. After all, experimental calculations are applied on a nonlinear differential equation and the results compared with interval Euler and interval Taylor methods to show the system efficiency.

REFERENCES

1. "*IEEE Approved Draft Standard for Interval Arithmetic (Simplified)*," IEEE P1788.1/D9.9, June 2017, (2017), 1-36.
2. G. Alefeld and G. Mayer, "*Interval analysis: theory and applications*," Journal of computational and applied mathematics, 121 (1), (2000), 421-464.
3. J. Campos, E. Assunção, G. Silva, W. Lodwick, and M. Teixeira, "*Discrete-time interval optimal control problem*," International Journal of Control, (2017), 1-7.
4. W. Edmonson and G. Melquiond, "IEEE Interval Standard Working Group - P1788: Current Status," in Proc. Name, (2009), 231-234.
5. M. M. Hosseini, "*Adomian decomposition method with Chebyshev polynomials*," Applied Mathematics and Computation, 175 (2), (2006), 1685-1693.

6.N. Razmjooy and M. Ramezani, "*Optimal Control of Two-Wheeled Self-Balancing Robot with Interval Uncertainties Using Chebyshev Inclusion Method*," Majlesi Journal of Electrical Engineering, 12 (1), (2018), 13-21.
7.N. Razmjooy and M. Ramezani, "*Solution of the Hamilton jacobi bellman uncertainties by the interval version of adomian decomposition method*," Int Rob Auto J, 4 (2), (2018), 113-117.