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Hop domination polynomial of graphs

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ABSTRACT

Let G be a simple graph of order n. The hop domination polynomial of G is the polynomial $D_h(G, x) = \sum_{i=\gamma_h(G)}^{|V(G)|} d_h(G, i)x^i$, where d_h (G, i) is the number of hop dominating sets of G of size i and $\gamma_h(G)$ is the hop domination number of G. In this paper we study D_h (G, x) of a graph. We classify many families of graphs by studying their hop domination polynomial.

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1 INTRODUCTION

Let G=(V,E) be a simple graph. The order of G is the number of vertices of G. For any vertex $\upsilon \in V$, the open neighborhood of υ is the set $N(\upsilon) = \{u \in V | u\upsilon \in E\}$ and the closed neighborhood of υ is the set $N[\upsilon] = N(\upsilon) \cup \{\upsilon\}$. For a set S⊆V, the open neighborhood of S is $N(S) = \bigcup_{\upsilon \in S} N(\upsilon)$ and the closed neighborhood of S is N[S] = N(S)∪ S. A set S ⊆ V is a dominating set if N[S]=V, or equivalently, every vertex in V \ S is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality od a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The family of all γ-sets of a graph G is denoted by $\Gamma(G)$. The distance between two vertices u and υ, denoted by $d(u, \upsilon)$ is the length of the shortest path between u and υ. A subset D ⊆ V(G) is a hop domination set of G if for every $\upsilon \in V\setminus D$ there exists a $u \in D$, such that $d(u.\upsilon)$=2, the vertex u is called a hop vertex. The hop domination number of G, denoted by $\gamma_h(G)$ is the minimum cardinality of a hop dominating set of G. For more see (Ayyaswamy et.al., 2015. Haynes et al., 1998. Natarajan and Ayyaswamy., 2015).

The i-subset of V(G) is a subset of V(G) of size i. Let D(G, i) be the family of dominating sets of a graph G with cardinality i and let d(G, i) =$|D(G.i)|$. The dominating polynomial D(G,x) of G is defined as D(G, x) = $\sum_{i=\gamma(G)}^{|V(G)|} d(G, i)\, x^i$, where $\gamma(G)$ is the domination number of G. The family of all dominating sets of G with cardinality i containing a vertex υ is denoted by D_υ (G,i), and $d_\upsilon(G, i)$=$|D_\upsilon(G, i)|$, see (Alikhani and Peng., 2011). The star $K_{1.n-1}$ has one vertex υ of degree n-1 and n-1 vertices of degree one.

A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Likewise, a cycle on three or more vertices is a simple graph whose vertices can be arranged in a cycle sequence in such a way that two vertices are adjacent if they are consecutive un the sequence, and are nonadjacent otherwise. The length of a path or a cycle is the number of its edges. A path or cycle of length k is called k-path or k-cycle.

A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m.n}$. The complement G^c of a simple graph G is the simple graph with vertex set V, such that two vertices are adjacent in G^c if and only if they are not adjacent in G.

By starting with a disjoint union of two graph G and H and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denoted G ⋁ H. The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a wheel with n spokes and denoted W_n.

The notion of generalized Peterson graph is introduced as follows: given integers $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$, the graph GP(n,k) is defined on the set $\{x_i . y_i | i \in \mathbb{Z}_n\}$ of 2n vertices, with the edges given by $x_i x_{i+1}$, $x_i y_i$, $y_i y_{i+k}$ for all i.

In this notation, the Peterson graph is GP (5,2), we denote the Petersen graph by P=P(5,2). For more information, refer to (Biggs., 1993. Bondy and Murty., 2008). We use the notation $\binom{n}{r} = \frac{p(n.r)}{r!} = \frac{n!}{r! \times (n-r)!}$.

It has been already studies the domination, the connected domination polynomials of graph and edge domination, connected edge domination polynomials of graphs, see (Alikhani and Peng., 2011) Chaluvaraju and Chaitra., 2016. Mojdeh and Emadi., 2018).

In this paper, we study the hop dominating sets and hop domination polynomial of any graph in particular we classify the hop domination polynomial of star, complete bipartite graph $K_{m.n}$, Petersen graph P(5,2).

2 THE HOP DOMINATION POLYNOMIAL OF GRAPHS

In this section, we study the hop domination sets and hop domination polynomial of graphs. In particular star graph, complete graph K_n and complete bipartite graph $K_{m,n}$.

Observation 1. For any graph G of order n we have,

i) $d_h(G, n) = 1$.

ii) $d_h(G, n-1) = n - i$. where i is the number of vertices of degree $n-1$.

iii) If G is not K_1, then $d_h(G, 1) = 0$.

Proof. (i) and (iii) are clear.

ii) If G has the vertices of degree n-1, every hop dominating set of G with cardinality n-1 should include such vertices. So the proof is complete.

Lemma 2. Let G be a connected graph of order n. If G has exactly k subsets of two vertices like {u,v} such that the vertices u, v are only vertices with d(u, v)=2 and no vertex is in distance 2 of them, then $d_h(G, n-2) = \binom{n}{n-2} - k$.

Proof. Every set of $n-2$ vertices is a hop dominating set, except that one obtain from deleting a vertex and only itself hop vertex. $d_h(G, n-2) = \binom{n}{n-2} - k$.

Lemma 3. (Proposition 1, Natarajan and Ayyaswamy., 2015)

i) For a complete graph Kn, $\gamma_h(K_n) = n$.

ii) For a complete bipartite graph $K_{m,n}$, $\gamma_h(K_{m,n}) = 2$.

ii) For a path P_n on n vertices,

$$\gamma_h(P_n) = \begin{cases} 2r & n = 6r \\ 2r+1 & n = 6r+1 \\ 2r+2 & n = 6r+s; 2 \leq s \leq 5 \end{cases}$$

iv) For a cycle C_n of length n,

$$\gamma_h(C_n) = \begin{cases} 2r & n = 6r \\ 2r+1 & n = 6r+1 \\ 2r+2 & n = 6r+s; 2 \leq s \leq 5 \end{cases}$$

v) $\gamma_h(W_n) = 3$ where W_n is a wheel with n-1 spoke.

vi) $\gamma_h(P) = 2$ where P denotes the Petersen graph.

Now as an immediate consequence from Observation 1, Lemma 2 and Lemma 3 we have;

Corollary 4. i) For every n ≥4, $d_h(P_n, n-1) = n$ and $d_h(P_3, 2) = 2$.
ii) For every n ≥ 6, $d_h(P_n, n-2) = \binom{n}{n-2} - 4$, $d_h(P_4, 2) = \binom{4}{2} - 2$ and $d_h(P_5, 3) = \binom{5}{3} - 3$.

Corollary 5. i) For every n ≥ 4, $d_h(C_n, n-1) = n$.
ii) For every n ≥ 5, $d_h(C_n, n-2) = \binom{n}{n-2}$ and $d_h(C_4, 2) = \binom{4}{2} - 2$.

Corollary 6. Let G be a graph K_n or K_n^c. $D_h(G, x) = x^n$.

We also have the nice similar results for complete bipartite graphs two next results have straightforward for proof and they are left.

Theorem 7. Let G be a graph $K_{1.n-1}$. $n \geq 3$,
i) $d_h(G, n-i) = \binom{n-1}{n-i-1}$. $0 \leq i \leq n-2$.
ii) $D_h(G, x) = \sum_{i=0}^{n-2} \binom{n-1}{n-i-1} x^{n-i}$.

Theorem 8. For a complete bipartite graph $K_{m.n} (2 \leq m \leq n)$, we have;

$$D_h(K_{m.n}, x) = ((1+x)^n - 1)((1+x)^m - 1)$$

Proof. Suppose D is a hop dominating set with cardinality i, we have 1 < i < n+m
and

$$d_h(K_{n.m}, i) = \begin{cases} \binom{m+n}{i} & i > n \\ \binom{m+n}{i} - \binom{m}{i} - \binom{n}{i} & 2 \leq i \leq m \\ \binom{m+n}{i} - \binom{n}{i} & m < i \leq n \end{cases}$$

Since coefficient x^i in the hop domination polynomial is the same as $d_h(K_{m,n}, i)$,
Therefore $D_h(K_{m.n}, x) = ((1+x)^n - 1)((1+x)^m - 1)$.

3 HOP DOMINATION POLYNOMIAL OF THE PETERSEN GRAPH P(5, 2)

In this section we shall investigate the hop domination polynomial of the Petersen graph.

We know $\gamma_h(P) = 2$, $d_h(P, 2) = 15$.

These hop dominating sets are listed below by Figure 1.
$D_h(P,2)$={{1,2},{1,5},{1,7},{2,3},{2,8},(3,4},{3,9},{4,5}{4,10},{5,6},{6,8},
{6,9},{7,9},{7,10},{8,10}}.

Note that we denote vertices a_1, a_2, a_3, a_4, a_5 for outer 5-cycle and vertices b_1, b_2, b_3, b_4, b_5 for inner 5-cycle.

Lemma 9. For the Petersen graph P, $d_h(P; 3) = 110$.

Proof. Each set of three vertices is hop dominating set, except that one obtain from deleting a vertex and their six hop vertices. Therefore $d_h(P, 3) = \binom{10}{3} - 10$.

Lemma 10. For the Petersen graph P, $d_h(P, j) = \binom{n}{j}$, $4 \leq j \leq 10$.

Proof. Since for every vertex like a1 in the Petersen graph there exists six vertices
a_3, a_4, b_2, b_3, b_4, b_5 such that $d(a_1, a_i) = d(a_1, b_j) = 2$, i = 3, 4, j = 2,3, 4, 5, the proof
is complete.

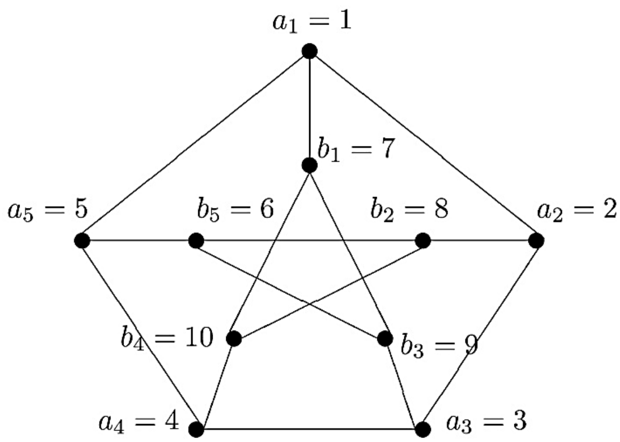

Figure **1. Petersen graph with label**

By Lemma 9 and Lemma 10, we have the main result.

Theorem 11. The hop domination polynomial of the Petersen graph is:
$D_h(P, X) = X^{10}+10X^9+45X^8+120X^7+210X^6+252X^5+210X^4+110X^3+15X^2$.

CONCLUSIONS AND QUESTION

We end the manuscript with some questions. It has been already studied total domination polynomial, connected domination polynomial and independent domination polynomial. Here we studied the hop domination polynomial of graphs. There may be posed the questions: whether we can study the total (connected, independent) domination polynomial? Also it has been studied the roots of domination polynomial of graphs and characterize the graphs by their domination polynomial roots. Here there may be posed, whether we characterize the graph by the roots of their hope domination polynomials?

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