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On Connectivity of Basis Graph of Splitting Matroid

Moein Pourbaba, Habib Azanchiler
Faculty of science/Urmia University
Urmia University, Urmia, Iran
m.pourbaba@urmia.ac.ir; h.azanchiler@urmia.ac.ir

ABSTRACT

For a set $\mathcal{C}$ of circuits of a matroid M, $B(M.\mathcal{C})$ is defined by the graph with one vertex foreach basis of M, in which two bases B_1 and B_2 are adjacent if $B_1 \cup B_2$ contains exactly one circuit and this circuit lies in $\mathcal{C}$. For two elements of a and b of ground set of a binary matroid M a splitting matroid $M_{a.b}$ is constructed. It is specified by two collections of circuits $\mathcal{C}_0$ and $\mathcal{C}_1$ dependent with collections of circuits of M. We want to study connectivity of $B\left(M_{\{a.b\}}.\mathcal{C}_0\right)$.

KEYWORDS: matroid, basis graph, splitting matroid, connected matroid

1 INTRODUCTION

We assume that reader is familiar with the basic concepts of matroid theory and graph theory. For more details one can see [7] for matroid theory and [4] for graph theory. The basis graph of a matroid M is the graph $B(M)$ whose vertex set is the set of bases of M and two bases B_1 and B_2 are adjacent if the size of symmetric difference is two. In other words, two bases are adjacent if they differ in only one element. The tree graph of a graph G, $T(G)$, is the basis graph of the matroid of $M(G)$. Cummins [1] proved that $T(G)$ is hamiltonian and therefore connected. The analogue result for matroids was proved by Holzmann and Harary [5].

Several variations of the basis graph have been studied, for instance, Li et al. [6] defined the basis graph of a matroid M by a set of circuits $\mathcal{C}$, $B(M.\mathcal{C})$, as the subgraph of $B(M)$ such that two basis B_1 and B_2 are adjacent if the unique circuit of M contained in the union $B_1 \cup B_2$ is a circuit of $\mathcal{C}$. They found a sufficient condition and a necessary condition for this graph to be connected when M is a binary matroid, and they proved that given an element e of a binary matroid M, $B(M.\mathcal{C}_e)$ is connected if $\mathcal{C}_e$ is the family of all the circuits that contains e. Figueroa et al. [2] generalized this result and they proved that it is true for every matroid of M with given $\mathcal{C}_e$.

The splitting operation for a graph is defined by the following way [3]; let G be a graph and v a vertex in which d $d(v) \geq 3$. Let $a = vv_1$ and $b = vv_2$ are two edges incident at v, then splitting away the pair a, b from v results in a new graph $G_{\{a.b\}}$ obtained from G by deleting the edges a and b, and adding a new vertex v' adjacent to v_1 and v_2. The transition from G to $G_{\{a.b\}}$ is called the *splitting operation* on G. We also denote the new edges $v'v_1$ and $v'v_2$ in $G_{\{a.b\}}$ by a and b, respectively.

The notion of the splitting operation extends to binary matroid in the following way [8]. Let $M = (E.\mathcal{C})$ be a binary matroid and a , b be two elements of E . Let

$$\mathcal{C}_0 = \{C \in \mathcal{C} : \ a.b \in \ C \ or \ C \in \mathcal{C} : \ a.b \ \notin \ C\}$$

$$\mathcal{C}_1 = \{C_1 \cup \ C_2 : C_1 . C_2 \in \mathcal{C}. \ C_1 \cap \ C_2 = \emptyset. \ a \in \ C_1. \ b \in \ C_2 \ and \ C_1 \cup \ C_2. \text{ contains no member of } \mathcal{C}_0 \}$$

Let $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$, then $(M_{\{a.b\}}.\mathcal{C}')$ is a binary matroid. As the collection of cycles of splitting graph is the same with the collection of circuits of splitting binary matroid for which defined above, we used this notation. In [8], the authors showed that $M_{\{a.b\}}$ obtains from M by adding an extra row to binary matrix representation of M , in which arrays respect to a , b are 1 and remind arrays are 0 .

Shikare and Azadi [9] characterized the collection of bases of a splitting binary matroid as the following theorem.

Theorem 1.1. Let $M = (E.\mathcal{C})$ be a binary matroid and $a.b \in E$. Let B be the set of all bases of M . Let

$\mathcal{B}_{\{a.b\}} = \{B \cup \{\alpha\}: B \in \mathcal{B}. \ \alpha \in \ E - B \ \ and\ the\ unique\ circuit\ contained\ in\ B \cup \{\alpha\} \ \ contains\ either\ a \ \ or \ \ b\}$. Then $\mathcal{B}_{\{a.b\}}$ *is a set of bases of* $M_{\{a.b\}}$.

2 MAIN RESULTS

We want to consider two cases of $B(M_{\{a.b\}}.\mathcal{C})$ respect the two collections of circuits of $M_{\{a.b\}}$. First, we start with a helpful lemma.

Lemma 2.1. Let M be a binary matroid and $\{a.b.c\}$ be a triangle of M . Let B_1 be a basis of M contains two elements of this triangle and $B_2 \subseteq E(M)$ such that $|B_2| = |B_1|$ and B_2 in comparison with B_1 is just differ in one element and this element belongs to the triangle, then B_2 is a basis of M .

Proof. Let $B_1 = \{a.b.1.2.\dots.n\}$ be a basis of M and $B_2 = \{a.t.1.2.\dots.n\}$ be assumed subset of the lemma. It is clear that $X = \{a.1.2.\dots.n\}$ is independent, we prove $X \cup \{t\} = B_2$ is too. Assume the contrary and let $X \cup \{t\}$ contains a circuit C . Clearly $t \in C$. If $a \in C$, as M is binary and $\{a.b.t\}$ is a triangle, then $a + b = t$. So $C' = b \cup C - \{t.a\}$ is a circuit and $C' \subseteq B_1$, a contradiction. If $a \notin C$, then $C'' = \{a.b\} \cup C - \{t\}$ is a circuit and $C'' \subseteq B_1$, a contradiction. Thus $X \cup \{t\}$ is independent and therefore B_2 is a basis of M . □

The following theorem is our main result.

Theorem 2.2. Let M be a binary matroid and $T = \{a.b\}$, where $a.b \in E(M)$. Let $\mathcal{C}_0$ be the collection of circuits in M in which meet T at even elements, then a sufficient condition for $B(M_{\{a.b\}}.\mathcal{C}_0)$ to be connected is that T lies in a triangle in M .

Proof. If $\mathcal{C}_1$ at the collection of circuits of splitting matroids is empty set, then $M_{\{a.b\}} = M$ and hence $\mathcal{C}_0 = \mathcal{C}(M)$. Therefore $B(M.\mathcal{C}_0) = B(M)$ and conclusion holds. Thus, suppose $\mathcal{C}_1$ is non-empty set. Let $\{a.b.t\}$ be the assumed triangle. We consider four cases about vertices of $B(M_{\{a.b\}}.\mathcal{C}_0)$. In these cases, B_i are bases of $M_{\{a.b\}}$ as characterized in Theorem 1.1 and $x.y \in E(M_{\{a.b\}}) - (T \cup t)$.

case 1. $B_1 = \{a.1.2.\dots.x\}$ and $B_2 = \{a.1.2.\dots.y\}$.

Since $M_{\{a.b\}}$ does not have a circuit with just an element of T , then $B_1 \cup B_2$ contains a circuit C that avoids a . Hence $C \cap T = \emptyset$, so $C \in \mathcal{C}_0$. We conclude B_1 and B_2 are adjacent in $B(M_{\{a.b\}}.\mathcal{C}_0)$.

case 2. $B_1 = \{a.1.2.\dots.x\}$ and $B_2 = \{a.1.2.\dots.b\}$.

Suppose B_1 and B_2 are not adjacent. Then $t \notin B_1 \cup B_2$; otherwise the triangle $\{a.b.t\}$ contained in $B_1 \cup B_2$, contradicting the fact that B_1 and B_2 are not adjacent. By Lemma 2.1 $B_3 = \{a.1.2.\dots.t\}$ is a basis of $M_{\{a.b\}}$. Now $B_2 \cup B_3$ contains a unique circuit in which t and b belong to it and since a belongs to that union, the circuit is the triangle. Thus, it is on $\mathcal{C}_0$. Then B_2 and B_3 are adjacent. By the first case B_1 and B_3 are adjacent. Then there is a path from B_1 to B_2 .

case 3. $B_1 = \{a.b.1.2.\dots.x\}$ and $B_2 = \{a.b.1.2.\dots.y\}$.

If a circuit X in $B_1 \cup B_2$ is a member of $\mathcal{C}_0$, the result is trivial and these two vertices are adjacent in $B(M_{\{a.b\}}.\mathcal{C}_0)$. By the Lemma 2.1, $B_3 = \{a.t.1.2.\dots.x\}$ and $B_4 = \{a.t.1.2.\dots y\}$ are two bases of $M_{\{a.b\}}$. By the second case $B_1 = \{a.b.1.2.\dots.x\}$ and $B_3 = \{a.t.1.2.\dots.x\}$ are adjacent and similarly two bases of $B_2 = \{a.b.1.2.\dots.y\}$ and $B_4 = \{a.t.1.2.\dots.y\}$ are adjacent by the second case. In the other hand B_3 and B_4 are adjacent by the first case. Therefore, there is a path between B_1 and B_2.

case 4. $B_1 = \{1.2.\dots.a\}$ and $B_2 = \{1.2.\dots.b\}$.

Suppose B_1 and B_2 are not adjacent in $B(M_{\{a.b\}}.\mathcal{C}_0)$. The uniqe circuit X in $B_1 \cup B_2$ is disjoint union of two circuits C_1 and C_2 of M such that each of them meets T precisely in one element. In fact, $C_1 = \{a.a_1.a_2.\dots.a_m\} \subseteq B_1$ and $C_2 = \{b.b_1.b_2.\dots.b_n\} \subseteq B_2$. It is clear that $t \notin B_1 \cup B_2$. We construct B_3 by the following way; we delete a member of C_1 like a_1 and add t in it. Without loss of generality, we can assume that $a_1 = 1$. Hence $B_3 = \{t.2.\dots.a\}$. Since $a + b = t$ and $B_3 - \{a.t\} \subseteq B_2$ it is clear that B_3 is a basis of $M_{\{a.b\}}$. Now by the second case B_1 and B_3 are connected by a path. Suppose $B_4 = \{b.2.\dots.a\}$ is constructed by deleting t and adding b. By the Lemma 2.1, B_4 is a basis of $M_{\{a.b\}}$. By the second case B_4 and B_3 are connected by a path too. Now if we apply the same procedure for the basis B_2 and without loss of generality by considering $b_1 = 2$, we get bases $B_5 = \{1.t.\dots.b\}$ and $B_6 = \{1.a.\dots.b\}$ that by the second case there is a path between B_2 to B_5 and a path between B_5 to B_6. Now notice that $B_4 \Delta B_6 = \{1.2\}$ and B_4 and B_6 are connected by a path by the second case. Hence there is a path between B_1 and B_2.

Now suppose that B_1 and B_2 are two arbitrary bases of $M_{\{a.b\}}$. As $M_{\{a.b\}}$ is connected, there is a path from B_1 to B_2 in $B(M_{\{a.b\}})$. Then B_1 has an adjacent vertex in $B(M_{\{a.b\}})$. These two vertices are connected with a path by using four above cases. Thus, we conclude B_1 and B_2 in $B(M_{\{a.b\}}.\mathcal{C}_0)$ are connected by a path, then the graph $B(M_{\{a.b\}}.\mathcal{C})$ is connected. $\square$

Evidently the sufficient condition in the theorem is not necessary.

Corollary 2.3. Let M be a matroid and $\mathcal{C}_0$ be specified collection of circuits in the last Theorem. Let S be a triangle of M. Suppose $T = \{a.b\}$ and T, S are disjoint. If $M_{\{x.y\}} = M_{\{a.b\}}$, where $x.y \in S$, then $B(M.\mathcal{C}_0)$ is connected.

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