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Solving an optimal control problem of a linear time-varying delay systems via cardinal Hermite functions

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ABSTRACT

In this paper, we present a new numerical method to solve an optimal control problem of a linear time varying delay systems. In this method, we use the cardinal Hermite functions as basis functions for approximation of functions. Also, we obtain operational matrices of integration and delay and use them to reduce the mentioned optimal control problem to a system of algebraic equations, which can be solved using newton's iteration method.

KEYWORDS: Optimal control, Delay systems, Cardinal Hermite functions.

1 INTRODUCTION

The control of systems with time-delay is important. The delays may appear in the system state and control vectors. Delays occur frequently in chemical processes, electronic, aerospace and mechanical systems and industrial processes.

Many orthogonal functions or polynomials, such as block-pulse functions [1,10],Walsh functions[7] Fourier series [3], Legendre polynomials [13], Chebyshev polynomials [5] and Laguerre polynomials [6], were used to derive solutions of some systems. Recently, the authors [8,9,12,11,14] defined the new basis functions bases on the different kinds of hybrid functions.

The present paper is organized as follows: In section 2, we introduce basis cardinal Hermit and present delay and integration operational matrices. In section 3, we apply a numerical method for solving under study problem. The numerical results are reported in section 4.

2 DEFINITION AND PROPERTIES OF CARDINAL HERMITE

The cardinal Hermite functions $\xi=(\xi_1(t),\xi_2(t),\xi_3(t),\xi_4(t))$ are defined as [4]:

*Speaker

$$
\begin{aligned}
&\xi_1(t) = (t-1)^4(1-4t+10t^2-20t^3)\chi_{[-1,0]}(t) + (t-1)^4(1+4t+10t^2+20t^3)\chi_{[0,1]}(t),\\
&\xi_2(t) = (t+1)^4(t-4t+10t^3)\chi_{[-1,0]}(t) + (t-1)^4(t+4t+10t^3)\chi_{[0,1]}(t),\\
&\xi_3(t) = (t+1)^4(\frac{t^2}{2}-2t^3)\chi_{[-1,0]}(t) + (t-1)^4(\frac{t^2}{2}+2t^3)\chi_{[0,1]}(t),\\
&\xi_4(t) = (t+1)^4(\frac{t^3}{6})\chi_{[-1,0]}(t) + (t-1)^4(\frac{t^3}{6})\chi_{[0,1]}(t),
\end{aligned} \tag{1}
$$

where

$$
\chi_{[t_0,t_1]} = \begin{cases} 1 & t \in [t_0, t_1] \\ 0 & Otherwise. \end{cases}
$$

The functions satisfy the following property:

If $m \in \mathbb{Z}$, we obtain:

$$
\xi(m) = \delta_m[1,0,0,0]^T, \quad \xi'(m) = \delta_m[0,1,0,0]^T,
$$

$$
\xi''(m) = \delta_m[0,0,1,0]^T, \quad \xi'''(m) = \delta_m[0,0,0,1]^T,
$$

where

$$
\delta_m = \begin{cases} 1, & m = 0 \\ 0, & otherwise. \end{cases}
$$

In this paper, we use the cardinal Hermite functions in[0,1] with $\xi_{m,k}(t) = \xi_m(2^M t - k),\ m = 1,..,4,\ k = 0,..,2^M$ and arbitrary positive integer M, Now we introduce the cardinal Hermite vector as [2]:

$$
\psi(t) = [\xi_{1,0}(t), \xi_{2,0}(t), \xi_{3,0}(t), \xi_{4,0}(t) \,|\, \ldots \,|\, \xi_{1,2^M}(t), \xi_{2,2^M}(t), \xi_{3,2^M}(t), \xi_{4,2^M}(t)]^T. \tag{2}
$$

We suppose that $A = \int_0^1 \psi(t)\psi^T(t)dt$, then we obtain

$$
A = \begin{bmatrix} R & H & 0 & \ldots & 0 \\ G & R_1 & H & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & G & R_{2^M-1} & H \\ 0 & & 0 & G & R \end{bmatrix}, \tag{3}
$$

where A is a block tridiagonal matrix with dimension $(2^M+1)4 \times (2^M+1)4$. The matrices R, G, H and $R_i, i = 1,...,2^M-1$ are block matrix with dimension 4×4.

2.1 Function approximation

An function $f \in L^2[0,1]$ can be expanded in term of cardinal Hermite functions as:

$$
f(t) \approx \sum_{k=0}^{2^M}\sum_{m=1}^{4} c_{m,k}\xi_{m,k}(t) = C^T\psi(t), \tag{4}
$$

where C is unknown coefficient vector with dimension $(2^M+1)4 \times 1$:

$$
C = [c_{1,0}, c_{2,0}, c_{3,0}, c_{4,0} \,|\, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1} \,|\ldots|\, c_{1,2^M}, c_{2,2^M}, c_{3,2^M}, c_{4,2^M}]^T.
$$

Thus, we have

$$C^T = F^T A^{-1}, \tag{5}$$

where F is vector with dimension $(2^M+1)4\times 1$ and can be achieved by

$$F=[\int_0^1 f(t)\xi_{1,0}(t)dt, \int_0^1 f(t)\xi_{2,0}(t)dt, \int_0^1 f(t)\xi_{3,0}(t)dt, \int_0^1 f(t)\xi_{1,0}(t)dt \,|\, ... \,|\, \int_0^1 f(t)\xi_{1,2^M}(t)dt, \int_0^1 f(t)\xi_{2,2^M}(t)dt,$$
$$\int_0^1 f(t)\xi_{3,2^M}(t)dt, \int_0^1 f(t)\xi_{1,2^M}(t)dt].$$

2.2 Operational matrices of delay and integration

We approximate $\int_0^t \psi(s)ds$ and $\psi(t-\upsilon)$ by the cardinal Hermite functions by giving a consideration to Eq. (5) the integral operational matrix $P=[p_{i,j}]$ can be obtained

$$\int_0^t \psi(s)ds \approx P\psi(t), \tag{6}$$

where

$$\int_0^t \xi_i(s)ds = \sum_{j=1}^{(2^M+1)4} p_{ij}\xi_j(t) \qquad i=1,...,(2^M+1)4.$$

Also, the delay function $\xi_i(t-\upsilon)$ can be approximated by the cardinal Hermite functions as

$$\xi_i(t-\upsilon) = \sum_{j=1}^{(2^M+1)4} d_{i,j}\xi_j(t), \qquad i=1,...,(2^M+1)4.$$

Then, the delay operational matrix $D=[d_{i,j}]$ is defined as

$$\psi(t-\upsilon) \approx D\psi(t), \qquad t>\upsilon, 0\le t\le t_f. \tag{7}$$

The dimension of these operational matrices is $(2^M+1)4\times(2^M+1)4$.

3 PROBLEM STATEMENT

We consider the linear time-varying delay system as [14]

$$\min J = \frac{1}{2}X^T(t_f)SX(t_f) + \frac{1}{2}\int_0^{t_f}[X^T(t)Q(t)X(t) + U^T(t)R(t)U(t)]dt, \tag{8}$$

$$\begin{aligned} &\dot{X}(t) = W(t)X(t) + K(t)X(t-\upsilon) + G(t)U(t), \quad 0\le t\le t_f,\\ &X(0) = X_0,\\ &X(t) = \phi(t) \qquad -\upsilon \le t < 0, \end{aligned} \tag{9}$$

where

$$X(t) = [x_1(t), x_2(t), ..., x_n(t)]^T,$$

$$U(t) = [u_1(t), u_2(t), ..., u_m(t)]^T.$$

$W(t), K(t), G(t), Q(t), R(t)$ are known matrix functions. X_0 is a constant specified vector and $\phi(t)$ is given function. The aim is to find the optimal control $U(t)$ and the corresponding state trajectory $X(t)$ while satisfying in Eq. (8) and Eq. (9).

For this purpose, we expand each components of the state and control vectors and also the elements of matrices $W(t), K(t), G(t)$ and $X_0, \phi(t)$ using the cardinal Hermite functions.

Each component can be expanded as

$$x_i(t) = X_i^T \psi(t),$$

$$u_j(t) = U_j^T \psi(t),$$

which X_i, U_j are unknown coefficient vectors. We consider $\psi_1(t) = I_n \otimes \psi(t), \psi_2(t) = I_m \otimes \psi(t),$ where I_n, I_m are identity matrices, then we have

$$X(t) = \psi_1^T(t) X, \qquad (10)$$
$$U(t) = \psi_2^T(t) U.$$

Also, the matrices $W(t), K(t), G(t)$ and X_0 can be expanded as follow:

$$W(t) = [W_{10}, W_{11}, ..., W_{12^M}, ..., W_{40}, W_{41}, ..., W_{42^M}]^T \psi_1(t) = W^T \psi_1(t),$$

$$K(t) = [K_{10}, K_{11}, ..., K_{12^M}, ..., K_{40}, K_{41}, ..., K_{42^M}]^T \psi_1(t) = K^T \psi_1(t), \qquad (11)$$

$$G(t) = [G_{10}, G_{11}, ..., G_{12^M}, ..., G_{40}, G_{41}, ..., G_{42^M}]^T \psi_2(t) = G^T \psi_2(t).$$

$$X_0 = \psi_1(t) F.$$

Now, to find approximate delay function $X(t-\upsilon)$ using cardinal Hermite functions, we set

$$\phi(t-\upsilon) = \psi_1^T(t) E,$$

$$X(t-\upsilon) = \begin{cases} \phi(t-\upsilon), & 0 \le t \le \upsilon, \\ X(t-\upsilon), & \upsilon \le t \le t_f. \end{cases} \qquad (12)$$

Then by using delay operational matrix we obtain

$$X(t-\upsilon) = \psi_1^T(t-\upsilon) X = \psi_1^T(t) \hat{D} X \quad \upsilon \le t \le t_f \Rightarrow X(t-\upsilon) = \begin{cases} \psi_1^T(t) E, & 0 \le t \le \upsilon, \\ \psi_1^T(t) \hat{D} X, & \upsilon \le t \le t_f. \end{cases} \qquad (13)$$

where $\hat{D} = I_n \otimes D$.

Also, if V is an arbitrary vector, we approximate the product of cardinal Hermite functions as

$$V^T \psi_i(t) \psi_i^T(t) \approx \psi_i^T(t) \tilde{V} \quad \text{i=1,2,} \qquad (14)$$

and we use it in following relations

$$W(t) X(t) = W^T \psi_1(t) \psi_1^T(t) X = \psi_1^T(t) \tilde{W}^T X,$$

$$G(t) U(t) = G^T \psi_2(t) \psi_2^T(t) U = \psi_2^T(t) \tilde{G}^T U,$$

similar to Eq. (13) and using Eq. (14) the following approximation is existed :

$$\int_0^t K(s) X(s-\upsilon) ds = \begin{cases} \psi_1^T(t) \hat{P}^T \tilde{K}^T E, & 0 \le t \le \upsilon, \\ \psi_1^T(t) Z \tilde{K}^T E + \psi_1^T(t) \hat{P}^T \tilde{K}^T \hat{D}^T X, & \upsilon \le t \le t_f, \end{cases} \qquad (15)$$

where $\hat{P} = I_n \otimes P$ is integral operational matrix and $\int_0^\upsilon \psi_1^T(t) dt = \psi_1^T(t) Z.$

Therefore, using Eqs.(10)-(15) and integrating Eq. (9), we obtain:

$$\int_0^t \dot{X}(t)dt = X(t) - X(0) = \psi_1^T(t)X - \psi_1^T(t)F = \psi_1^T(t)\hat{P}^T\tilde{W}^T X + \psi_1^T(t)\hat{P}^T\tilde{K}^T E + \psi_1^T(t)Z\tilde{K}^T E$$
$$+ \psi_1^T(t)\hat{P}^T\tilde{K}^T\hat{D}^T X + \psi_2^T(t)\hat{P}^T\tilde{G}^T U,$$

which can be rewritten as

$$B = F + (-I + \hat{P}^T\tilde{W}^T + \hat{P}^T\tilde{K}^T\hat{D}^T)X + Z\tilde{K}^T E + \hat{P}^T\tilde{K}^T E + \hat{P}^T\tilde{G}^T U. \quad (16)$$

Moreover, by substituting the introduced approximations in cost function, we have

$$J = \frac{1}{2}X^T\psi_1(t_f)S\psi_1^T(t_f)X + \frac{1}{2}X^T\left[\int_0^{t_f}\psi_1(t)Q(t)\psi_1^T(t)dt\right]X + \frac{1}{2}U^T\left[\int_0^{t_f}\psi_2(t)R(t)\psi_2^T(t)dt\right]U.$$

Now, the delay optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find X and U so that $J(X,U)$ is minimized subject to the constraints in Eq. (16).

Let

$$\min\ J^*(X,U,\lambda) = J(X,U) + \lambda^T B, \quad (17)$$

where the vector λ represents the unknown Lagrange multipliers.
The optimality conditions for this problem yield us to the following system

$$\frac{\partial}{\partial X}J^*(X,U,\lambda) = 0,$$
$$\frac{\partial}{\partial U}J^*(X,U,\lambda) = 0,$$
$$\frac{\partial}{\partial \lambda}J^*(X,U,\lambda) = 0.$$

The above equations can be solved using Newton's iterative method.

4 NUMERICAL EXAMPLES

Example1: Consider the delay system with different delay in state and control [8]:

$$\min\ J = \frac{1}{2}\int_0^1\left[x^2(t) + \frac{1}{2}u^2(t)\right]dt, \quad (18)$$

Subject to:

$$\dot{x}(t) = -x(t) + x(t-\frac{1}{3}) + u(t) + \frac{1}{2}u(t-\frac{2}{3}), \quad (19)$$

$$x(t) = 1, \quad -\frac{1}{3} \le t \le 0,$$

$$u(t) = 0, \quad -\frac{2}{3} \le t \le 0.$$

First, we expand x and u using cardinal Hermite functions and calculate delay operational matrices for $\upsilon = \frac{1}{3}, \frac{2}{3}$. We define D1 and D2 as delay operational matrices and set:

$$\psi(t-\frac{1}{3}) = D_1\psi(t),$$
$$\psi(t-\frac{2}{3}) = D_2\psi(t),$$

by using Eq. (15) we have

$$\int_0^t x(s-\frac{1}{3})ds = \begin{cases} h_1^T\psi(t), & 0 \le t \le \frac{1}{3}, \\ h_3^T\psi(t) + X^T D_1 P\psi(t), & \frac{1}{3} \le t \le 1, \end{cases}$$

where

$$x(0) = h_3^T\psi(t).$$

Similar to the previous approximation, we have

$$\int_0^t u(s-\frac{2}{3})ds = \begin{cases} 0, & 0 \le t \le \frac{2}{3}, \\ U^T D_2 P\psi(t), & \frac{2}{3} \le t \le 1. \end{cases}$$

Now, by substituting these approximations in conditions Eq. (19), we have

$$B = X^T(I + P - D_1P) - U^T(P - \frac{1}{2}D_2P) - (h_1 + h_2 + h_3) \ . \tag{20}$$

Moreeever, for J in Eq. (17) and using Eq. (3) and Eq. (20), we have

$$J^* = \frac{1}{2}[X^T AX + \frac{1}{2}U^T AU] + \lambda[X^T(I + P - D_1P) - U^T(P - \frac{1}{2}D_2P) - (h_1 + h_2 + h_3)] \ .$$

We have solved this example with $M = 1$ and $M = 2$. Table 1 shows the numerical results for this example and compared with method in [8].

Table 1. Comparison of the approximate solution J with present method and method [8]

Methods	**J**
Method in [8]	
M=5	0.37311253
M=6	0.37311241
Present Method	
M=1	0.02536402
M=2	0.01213871

Example 2: We wish to find the control $u(t)$ which minimizes the objective functional [3]:

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt,$$

subject to

$$\dot{x}(t) = -x(t) + u(t) - \frac{1}{2}u(t - \frac{2}{3}), \qquad 0 \le t \le 1$$

$$x(0) = 1,$$

$$u(t) = 0, \qquad -\frac{2}{3} \le t \le 0.$$

In this example, we obtain the delay operational integral for $\upsilon = \frac{2}{3}$. Table 2 displays the approximate solutions obtained for M=1,2,3. Also, the comparison of various value of cost function of present method with method in Ref.[3] is shown.

Table2. Comparison of the approximate solution J with present method and method [3]

Methods	**J**
Method in [3]	
M=4	0.195494339136
M=5	0.195494339136
M=6	0.195494339136
Present Method	
M=1	0.037891641331
M=2	0.032999389385
M=3	0.028651598722

5 CONCLUSION

In this present paper, an efficient and simple method is presented to solve a wide class of optimal control of linear delay systems. We have utilized the cardinal Hermite functions as basis. .The integral operational matrix, together with the delay operational matrix are used to solve the optimal control of linear delay systems. By using them, the problem reduce to a set of algebratic equations. The achieved solutions with this method demonstrate that the method is efficient.

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