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ATOMS OF CYCLIC EDGE CONNECTIVITY IN TETRAVALENT GRAPHS

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ABSTRACT

A *cyclic edge-cut* of a connected graph *G* is an edge set, the removal of which separate two cycles. If *G* has a cyclic edge-cut, then it is *called cyclically separable.* An induced subgraph *P* of *G* is called a *cyclic part* of *G* if there exists a cyclic edge-cut *S* such that *P* is a component of *G — S.* A cyclic part minimal under inclusion is called an *atom.* Also a cyclic part with minimum cardinality is called *proper atom.* For a cyclically separable graph G, the *cyclic edge connectivity* of a graph G, denoted by $A_c(G)$, is the minimal cardinality over all cyclic edge cuts. In this article, we establish some properties of atoms and cyclic parts in tetravalent graphs.

KEYWORDS: atom, cyclic edge connectivity, tetravalent graph.

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1. INTRODUCTION:

Throughout the paper graphs are finite, simple and undirected unless explicitly stated otherwise.

For a graph G, an edge set F is a cyclic edge-cut if G — F is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two vertex-disjoint cycles. Lovasz characterized all multigraphs without two disjoint cycles. The characterization can also be found in [1]. We call those graphs which do have cyclic edge-cut cyclically separable.

For a cyclically separable graph G, the cyclic edge-connectivity of G, denoted by $A_C(G)$, is defined as the cardinality of a minimum cyclic edge-cut of G (see, for example, Plummer 1972). Cyclic edge-connectivity plays an important role in many classic fields of graph theory. Most of the previous works focus on using the value of $A_c(G)$ as a condition to conquer other problems such as in studying integer flow conjectures [18], Hamiltonian graphs [6], fullerene graphs [3] . In fact, $A_c(G)$ can also be used as a measure of network reliability. The classic measure of network reliability is the edge-connectivity A(G) and/or the vertex connectivity $\kappa(G)$. In general, the larger A(G) and/or $\kappa(G)$ are, the more reliable the network is. However, A(G) and $\kappa(G)$ are worst-case measures.

Set £(G) = min{d(X) | X induces a shortest cycle in G}, where d(X) is the number of edges with one end in X and the other end in V (G) — X. It was proved in Wang and Zhang that $A_c(G) < £(G)$ for any cyclically separable graph G (see [11]). A cyclically separable graph G is called cyclically optimal, in short, $A_c(G)$-optimal, if $A_c(G) = £(G)$, and super cyclically edge-connected, in short, super-A_c, if the removal of any minimum cyclic edge-cut of G results in a component which is a shortest cycle of G (see [16]).

For two vertex sets $X,Y \subseteq V$, $[X,Y]$ is the set of edges with one end in *X* and the other end in *Y*. G[X] is the subgraph of *G* induced by vertex set *X*, *X* is the complement of X, $\delta(X) = [X, X^C]$ is the edges between X and X^C in G. Also we set $d(X) = |\delta(X)|$. A minimum cyclic edge-cut of *G* is simply called a A_c *-edge cut.* It is clear for any Ac-edge cut *F* that *G* — *F* has exactly two components. Let *X* be a non-empty proper subset of *V*(G). If $\delta(X)$ is a A_c -edge cut of G, then *X* is called a *proper fragment* or A_c *-fragment* of G. A A_c -fragment with minimum cardinality is called a *proper atom* or A_c *-atom.*

The concepts of fragment and atom were first proposed by Mader [7] and Watkins [12], and their variations play an important role in studying various kinds of connectivity. The key to the studies lies in proving the disjointness of atoms. For more results we refer the reader to [17, 18].

2. PRELIMINARIES :

For a graph G, denote by *V*(G), *E*(G), *A(G)* and Aut(G) the vertex-set, edge-set, arc- set and the full automorphism group of G, respectively. A graph *G* is said to be *vertex- transitive, edge-transitive* or *arc-transitive* if aut(G) acts transitively on *V*(G), *E*(G) or A(G), respectively. For $x,y \in V(G)$, an P_{xy} is a path from *x* to *y*. For any subgraph *H* of *G* and $x \in V(G)$ we use *d* (x) to denote the number of vertices in *H* adjacent to x. Also if *d* (x) = 1 then we say that *x* is a singular vertex.

The following proposition is due to Lovasz.

Proposition *2.1.* (see *[5])* Let G be a connected tetravalent graph with cyclic edge- connectivity $A_c(G) = k$. Also let P be a nontrivial cyclic part of G. If P has two disjoint cycles then we have one of the following cases:

There is a vertex x such that P — x is a forest,

There are three vertices x,y,z such that G — {x,y,z} has no edges,

There is a vertex x such that G — x is a single cycle,

The complete graph K5.

For the proof of the following proposition see [14, P.46, Claim2]

Proposition *2.2.* Suppose that G is a connected vertex-transitive graph. Then there exists a partition *{X1,X2,* • • • ,Xm} of V(G), ujhere m > 2, such that *G[Xj] = G[X]* and Xi is a proper atom for i *=1, 2,* • • • ,m.

In the following result $\theta 2$ and $\theta 4$ denote the graph consisting of two vertices and two and four parallel edges, respectively. Also *362* is a graph with three vertices and six edges. In fact in this graph, each of *02* has intersect in one vertex with two other $\theta 2$.

Lemma *2.3.* Suppose that G is connected tetravalent graph. Then G has no cycle- separating edge-cut if and only if it is isomorphic to one of K5, $\theta 4$ or $3\theta 2$. In fact if C is a shortest cycle in G then 5G is a cycle-separating edge cut unless G is K5, $\theta 4$ or $3\theta 2$**.**

Proof. Clearly K5, $\theta 4$ and $3\theta 2$ havenot cycle-separating edge cut. For the converse, assume that C is the shortest cycle in G and that 5G isnot a cycle-separating cut. We show that G is isomorphic to one of these graphs. Let C be the subgraph of G induced by the set V — V(C). Clearly C is forest. Thus C is either empty or it contains an isolated vertex or two vertices of valency one. First suppose that C is empty. If g > 3 then we could find a cycle with length less than the length of *C*. Thus *g* = 2 and $G = \theta 4$.

Now suppose that *C* contains an isolated vertex u. Then *u* is adjacent to four vertices on *C* and at least two of its neighbors, say *x* and *y* have distanced g/4 on *C*. If *P* is the path between *x* and *y* then *uxPyu* is a cycle of length h. Clearly $g < h <$ g/4 + 2 and so g = 2. In this case $G = 392$. Also we note that if C contains two isolated vertices then C contains at least four vertices, say a,b,c,d. If Q is the path between a and b on C then the length of $uaQbu$ is at least g/4 + 2. Thus $g <$ g/4 + 2 and so g = 2, a contradiction. Finally suppose that C has two vertices of valency one. Then we need at least 3 vertices m, n and p on C. Let L be the path between n and p on C. Now the length of $fnLp$ is at least g/3 + 2. Therefore $g < g/3 + 2$ and so g = 3. If C has more than two vertices then C must contain at least 4 vertices. Thus g = 2, a contradiction. Therefore, $C' \cong K2$ and so $G \cong$ K5.

Let g denote the girth of a graph G. By Lemma 2.3 we obtain the following result.

Lemma 2.4. Suppose that G is connected tetravalent graph. Then $A_c(G) < 2g$.

3. MAIN RESULTS:

Lemma *3.1.* Let G be a connected tetravalent graph. Also let P be a cyclic part of G. Then G doesnot have a singular vertex.

Proof. Suppose that P is a cyclic part of G which contains a singular vertex, say u. Let v be adjacent to u and v E P. Since G is tetravalent graph, we may assume that $w1,w2$ and $w3$ are three vertices adjacent to u in $G — P$. Assume that $e1$ = uw**1**, $e2 = uw2$ and $e3 = uw3$. Clearly, $e1,e2,e3 \in \delta P$ and $P' = P — u$ again has a cyclic. Therefore P is a cyclic part and $\backslash \delta P' \backslash = \backslash \delta P\backslash$ — 2. This violate our assumption.

Lemma *3.2.* Let G be a connected tetravalent graph. Then each non-trivial cyclic part or atom of G contains two vertices with degree more than 2.

Proof. Suppose that P is a non-trivial cyclic part or atom of G. Thus P contains at least one vertex with degree more than 2, say u. If u has degree 3 then has at least another vertex with degree 3, and so the assertion is hold. Thus we may suppose that u has degree 4. Thus P contains two cycles $C1$ and $C2$ such that $\delta (C1) < \delta P$ and $\delta (C2) < \delta P$, a contradiction. Therefore P has at least two vertices with degree more than 2.

Lemma *3.3.* Let G be a connected tetravalent graph. Then each cyclic part of G is connected. Also if $A_c(G) > 5$ then each cyclic part is a block. Moreover if $A_c(G) > 3$ then each atom is a block.

Proof. Let P be a cyclic part of X. If P is disconnected then it has a component Q contains a cycle. Note that $\delta Q \subset \delta P$. If $P = Q$ then $\backslash \delta Q\backslash < \backslash \delta P\backslash$. Therefore δQ is a cycle separating cut and $\backslash \delta Q\backslash < \backslash \delta P\backslash$, a contradiction.

Now suppose that P is a cyclic part of G containing a cut vertex, say v. Also suppose that P', P'' are two subgraphs of P such that $V(P') \cap V(P'')$ = {v} and $E(P') \cup E(P'')$ = E(P). We know that the maximum valency of P is at most 4. Thus we have the following cases for v.

Case 1. v has two neighbors in P and two neighbors in P .

We claim that always we could find two subgraphs of P such that contain a cycle. Since P is a cyclic part, it follows that at least one of the P or P has a cycle. If both of P and P have a cycle then the assertion is hold. Thus we may suppose that one of them has a cycle, say P . Now we consider the subgraph $P — \{e1,e2\}$ of G where $e1 = vx$, $e2 = vy$ and $x,y \in P'$. Clearly, $P — \{e1,e2\}$ has two components $P1$ and $P2 = P$. Also we note that $P2$ has a cycle. If x and y have degree at least 3 in P then x and y have degree at least 2 in P1. By Lemma 3.1 we know that each vertex of P has degree at least 2. Therefore each vertex of $P1$ also has degree at least 2, and so $P1$ contains a cycle.

Now suppose that one of vertices x or y has degree less than 3 in P. Without loss of generality we may suppose that $d(x) < 3$. Considering the subgraph $V(P1)$ — {x}. We see that all vertices of V(P1) — {x} have degree at least two and so contains a cycle.

Finally suppose that both of *x* and *y* have degree less than 3. Thus \ δ *P*\ > \ δ *P2*\ — 2 + 4 =| δ *P2*\ + 2 and so \ δ *P*\ > \ δ *P2*\. Since *P2* has a cycle, it implies that δ *P2* is a cycle separating cut and \ δ *P2*\ < $A_c(G)$, a contradiction. Now the assertion is hold and *P* has two subgraphs contain a cycle, say *Q1* and *Q2*. We have \4Q1\ + \ δ *Q2*\ = \ δ *P*\ + 4 = $A_c(G)$ + 4. Since both of *Q1* and *Q2* contain cycles, it implies that \ δ *Q1* \ > $A_c(G)$ and \ δ *Q2*\ > $A_c(G)$. Thus *2* $A_c(G)$ < | δ Q1| + \ δ *Q2*\ < $A_c(G)$ + 4 and so $A_c(G)$ < 4. Also note that if *P* is an atom then 2 $A_c(G)$ + 2 < \ δ Q1\ + \ δ *Q2*\ < $A_c(G)$ + 4 and so $A_c(G)$ < 2. Now in this case the proof is complete.

Case 2. *v* has three neighbors in one of *P* or *P* .In this case with the similar arguments as above we see that $A_c(G)$ < 4. Also if *P* is an atom then $A_c(G)$ < 2.

With the similar arguments in the proof of [9, Theorem 9] we have the following result.

Lemma *3.4.* Suppose that G is a connected tetravalent graph with edge cyclic connectivity $A_c(G)$ = k. If A and B are distinct cyclic parts of G with non-empty intersection then one of the following cases occurs:

(i) At least A ∩ B and A — B is a cyclic part of G contained in A.

(ii) Among the cyclic parts A, A^c, B and B^c at least one is a k-cycle.

Lemma *3.5.* Let G be a k-regular tetravalent vertex-transitive graph and k = *5*. Then all non-trivial proper atoms of G are cubic graph.

Proof. Suppose that *G* contains a non-trivial proper atom A. Also suppose that *x* and *y* are two vertices of A. By our assumption there is an automorphism *f* sending *x* to y. Clearly *A* ∩ *f* (A) ≠ ϕ . Thus *f* (A) = *A* and so *A* is vertex-transitive. Since *A* contains a 3-valent vertex, all the vertices of *A* are 3-valent, by vertex-transitivity. Thus *A* is a cubic graph.

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