A subclass of 2-derangements

Hossein Moshtagh[1]
Department of computer science, University of Garmsar, Garmsar, Iran
h.moshtagh@fmgarmsar.ac.ir

ABSTRACT

A derangement is a permutation that has no fixed points. In this paper, we study a subclass of derangements on n objects with two arbitrary forbidden objects location. We determine an explicit formula for the number of this subclass of derangements. we will prove that the Stirling transform of these numbers equals the binomial transform of the Bell numbers.

KEYWORDS: Derangement, Permutation, Stirling numbers, Bell numbers.

1 INTRODUCTION

A derangement over n integers $[n] = \{1,2,\ldots,n\}$ is defined as a permutation with no fixed points. The exercise of counting all derangements is a typical example of the inclusion-exclusion principle. The number of these permutations on S_n is denoted by $D(n)$. The inclusion-exclusion principle gives

$$D(n) = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!}. \quad (1)$$

The number of derangements is given by the recurrence relation

$$D(n) = \begin{cases} 0 & \text{if} \quad n = 1, \\ 1 & \text{if} \quad n = 2, \\ (n-1)(D(n-1) + D(n-2)) & \text{if} \quad n \geq 3. \end{cases}$$

To see this result and related results concerning derangement numbers, we refer the reader to [1].

Definition 1. 1. A subclass of derangements on $n + r$ objects such that none of the first r objects return to the first r objects location. Assume that $\Delta = \{1,2,\ldots,r\}$ and denote $\pi(\Delta)$ the action of a permutation $\pi \in S_{n+r}$ on the set Δ by defining $\pi(\Delta) = \{\pi(x),\ x \in \Delta\}$. A derangement $\pi \in S_{n+r}$ is called a Δ-derangement if $\pi(\Delta) \cap \Delta = \emptyset$. The number of Δ-derangements on S_{n+r} is denoted by $D_r(n)$.

It follows from the definition that n must be greater than or equal to r and it is equally easy to see that $D_1(n) = D(n+1)$, $D_2(2) = 4$ and $D_2(3) = 24$.

Definition 1. 2. Let us fix $r_i, s_i \in \mathbb{N}$ $(1 \leq i \leq k)$ with $r_i, s_i \leq n$. The permutation π of S_n that has no fixed points is called a k-derangement if $\pi(r_i) \neq s_i$ for $1 \leq i \leq k$. Let $D(k,n)$ denote the number of k-derangements of S_n. In other words, $D(k,n)$ counts the number of derangements with k forbidden positions. It follows from the definition that n must be greater than or equal to the maximum number in $\{r_i, s_i\colon 1 \leq i \leq k\}$. It is easy to see that $D(0,n) = D(n)$, $D(1,2) = 0$, $D(1,3) = 1$, $D(1,4) = 6$, $D(2,4) = 4$ and $D(2,5) = 24$.

[1] speaker

This paper is devoted to study a subclass of fixed point free permutations. We investigate the formula for counting the number of derangements with two arbitrary forbidden positions. We determine an explicit formula for the number of $D(2,n)$. Finally, we determine Stirling transform of derangement numbers.

2 PROPERTY OF 2-DERANGEMENTS

Definitions 1.1 and 1.2 are not the same. They also have different size for each $n \geq 4$. Only at $r = k = 2$, these definitions are the same. The main result of this paper are the following theorems.

Theorem ٢. 1. For all $n \geq 4$ we have that $D_k(n) = D(r, n-r)$ if and only if $r = k = 2$.

Corollary ٢. 2. For all $n \geq 4$ and $r < n-2$ we have that $D_r(n) < D(r, n-r)$.

Proof. Assume that $\Delta = \{1,2,\dots,r\}$. We take a permutation π on n elements which the first r objects to be members of Δ. If $\pi(\Delta) \cap \Delta = \emptyset$, then π is r-derangement and is Δ-derangement. Now suppose that B is an arbitrary subset of A such that $\pi \in S_{n+r}$ and $\pi(B) \cap B = \emptyset$. Then π is r-derangement but is not Δ-derangement. This complete the proof.

By Theorem 2.1, we obtain the following generalized recursive relation:

Theorem ٢. 3. For all $n \geq 4$ we have that

$$D(2,n) = D(1,n) - D(1,n-1).$$

Proof. Assume that

$$\mathcal{A} = \{\pi : \pi \in D(n), \pi(r) \neq s\}.$$

It is easy to see that $|A| = D(1,n)$. Now we count the set A in a different way for showing recursive relation. For this purpose, first assume that $\pi(s) = r$. In this case, since $\pi(i) \neq i$ for every $i \in [n]\backslash\{r\}$ and $\pi(r) \neq s$, the rest of the permutation π is a derangement on $n-1$ elements. Hence, we can choose this $n-1$ elements in $D(n-1)$ ways. This explains 1-derangement on $n-1$ elements. Now, consider the case $\pi(s) \neq r$. Then set $\Delta = \{r, s\}$. Since $\pi(s) \neq r$ and $\pi(r) \neq s$, so $\pi(\Delta) \cap \Delta = \emptyset$. In other words π is a Δ-derangement on $n-2$ elements. This case is counted by $D_2(n-2)$. By Theorem 2.1, $D_2(n-2) = D(2,n)$ and so $D(1,n) = D(1,n-1) + D(2,n)$. This complete the proof.

Theorem ٢. 4. Suppose $n > 3$ is integer. Then

$$D(2,n) = 2\sum_{i=2}^{n-2} (-1)^{n-2-i} \binom{i}{2} \frac{(n-2)!}{(n-2-i)!}.$$

Let $D_2(x)$ be defined as the exponential generating function of the sequence $\{D(2,n)\}_{n\geq 4}$. In other words,

$$D_2(x) = \sum_{n=4}^{\infty} D(2,n)\frac{x^n}{n!}.$$

Theorem ٢. 5. Suppose n is integer. Then

$$D_2(x) = \frac{2x^2}{(1-x)^3} e^{-x}.$$

Bell numbers count the number of partitions of a set of n distinguishable objects into non-empty subsets. The Bell numbers are denoted b_n, where n is an integer. Thus $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 15$ and by definition $b_0 = 1$. The exponential generating function of the Bell numbers is

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = e^{e^x - 1}. \quad (2)$$

Stirling numbers of the second kind numbers, denoted by $S(n,k)$, count the number of partitions of a set of n distinguishable objects into k non-empty subsets. In fact,

$$b_n = \sum_{k=1}^{n} S(n,k), \quad n \geq 1.$$

The Stirling transform is related to a couple of interesting series transformations. Namely, if

$$f(t)=\sum_{n=0}^{\infty}\frac{a_n}{n!}t^n$$
is an exponential generating function we have the series transformation formulas

$$f\left(\frac{\mu}{\lambda}(e^{\lambda t}-1)\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\sum_{k=0}^{n}S(n,k)\lambda^{n-k}\mu^k a_k\right\} \qquad (3)$$

By [2], we have the following exponential version of Euler's series transformation formula

$$e^{\lambda z}\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n=\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}a_k\right\} \qquad (4)$$

where λ is a parameter.

Theorem ٢. 6. Suppose n is an integer. Then

$$\sum_{k=0}^{n}S(n,k)(-1)^k D(2,n)=2\sum_{k=0}^{n}\binom{n}{k}b_k(-1)^{n-k}(1-2^{n-k+1}+3^{n-k}).$$

Proof. We use property (3) with $\lambda=1,\mu=-1$, replacing x in $D_2(x)$ by e^t-1

$$D_2(-(e^t-1))=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\sum_{k=0}^{n}S(n,k)(-1)^k D(2,n)\right\}.$$

By Theorem 2, property (2) and property (4) we have

$$\begin{aligned}D_2(-(e^t-1))&=2e^{-3t}(e^t-1)^2e^{e^t-1}.\\&=2(e^{-t}-2e^{-2t}+e^{-3t})e^{e^t-1}\\&=2(e^{-t}-2e^{-2t}+e^{-3t})\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\\&=2\sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}b_k(-1)^{n-k}(1-2^{n-k+1}+3^{n-k})\right\}.\end{aligned}$$

Comparing coefficients we find the equality of the theorem.

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