A Special Case of Derangement

R. Fallah-Moghaddam
Department of Computer Science, Garmsar University
P. O. BOX 3588115589, Garmsar, Iran
falah_moghaddam@yahoo.com

ABSTRACT

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. In other words, a derangement is a permutation that has no fixed points. In this article, we define a special case of derangement and we try to count the number of its members.

KEYWORDS: Derangement, Combinatorial method, Principle of inclusion and exclusion, Permutation.

1 INTRODUCTION

A derangement is a permutation in which none of the objects appear in their "natural" (i.e., ordered) place. Derangements are permutations without fixed points (i.e., having no cycles of length one). The derangement problem was formulated by P. R. de Montmort in 1708, and solved by him in 1713 (de Montmort 1713-1714). Nicholas Bernoulli also solved the problem using the inclusion-exclusion principle. The number of derangements of an n-element set is called the n-th derangement number or rencontres number, or the subfactorial of n and is sometimes denoted D_n. Counting the derangements of a set amounts to what is known as the hat-check problem, in which one considers the number of ways in which n hats can be returned to n people such that no hat makes it back to its owner. This number satisfies the recurrences

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Also, it is well-known that

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} = 0.3678 \ldots$$

The interesting this is that the number e itself also has applications in probability theory, in a way that is not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Then, for large n, the probability that the gambler will lose every bet is approximately $1/e$.

Recently, in [2], authors defined a special case of derangement as following:

Definition A. The FPF property obviously means that in this permutation any cycle is of length greater than one. What we add to this requirement is the following. We take a permutation on

$n + r$ letters and we restrict the first r of these to be in distinct cycles. We arrive at the definition of the subject of the paper An FPF permutation on $n + r$ letters will be called FPF r-permutation if in its cycle decomposition the first r letters appear to be in distinct cycles. The number of FPF r-permutations denote by $D_r(n)$ and call r-derangement number. The first r elements, as well as the cycles they are contained in, will be called distinguished. This definition was motivated by the extensive study of the so-called r-Stirling numbers of the first kind which count permutations with a fixed number of cycles where the same restriction on the first distinguished elements is added.

They proved that:

Theorem B. For all $n > 2$ and $r > 0$, we have

$$D_r(n) = rD_{r-1}(n-1)+(n-1)D_r(n-2)+(n+r-1)D_r(n-1)$$

Also, they find exponential generating function of $D_r(n)$ as following:

Theorem C. For any $r\epsilon\mathbb{N}$ for the exponential generating function of the sequence of r-derangements numbers we have that:

$$F_r(x) = \sum_{n=0}^{\infty} \frac{D_r(n)}{n!} x^n = \frac{x^r e^{-x}}{(1-x)^{r+1}}$$

Derangements are an example of the wider field of constrained permutations. For example, the ménage problem asks if n opposite-sex couples are seated man-woman-man-woman-... around a table, how many ways can they be seated so that nobody is seated next to his or her partner?

In following section, we define a new case of derangement and obtain some result about this special case of derangement.

MAIN THEOREM. Assume that A is a subset of $\{1,2,3,\dots,n\}$ and consider $\sigma\boldsymbol{\epsilon S_n}$is a derangement on A. Also assume that $|A| = m \leq n$ and D_m is the set of all derangement on A, then:

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n-k}{m-k} (m-k)!$$

For more result, see [1], [2], [3], [4] and [5].

2 MAIN RESULT

In this section we define a new special case of derangement and also we obtain some relation on this subset of derangement. This special case of derangement is a subset of block derangement.

Assume that $\sigma \epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\dots,n\}$. Consider that A is a subset of $\{1,2,3,\dots,n\}$. We say that σ is a derangement on A, if for any $\boldsymbol{i \epsilon A}$**,** we have $\sigma(i) \neq i$**.** Now, by this definition we have the following result.

MAIN THEOREM. Assume that A is a subset of $\{1,2,3,\dots,n\}$ and consider $\sigma \epsilon \mathrm{S_n}$is a derangement on A. Also assume that $|A| = m \leq n$ and D_m is the set of all derangement on A, then:

$$|D_m| = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n-k}{m-k} (m-k)!$$

Proof. We can use the inclusion-exclusion principle. Without loss of generality, we assume that $A = \{1,2,3,\dots,m\} \subseteq \{1,2,3,\dots,n\}$. Let $\sigma \epsilon \mathrm{S_n}$, $k \leq$m. Let A_k be the set of such elements of $\mathrm{S_n}$ such that $\sigma(i) = i$ for any $\boldsymbol{i \epsilon} B$, when $B \subseteq \{1,2,3,\dots,m\}$ with exactly k elements.

First, we prove that

$$\left| \bigcup_{1 \leq k \leq m} A_k \right| = \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \binom{n-i}{m-i} (m-i)!$$

By using inclusion-exclusion principle and the following relation the above formula is proved.

$$\left| \bigcap_{k \epsilon B \subseteq \{1,2,3,\dots,m\}} A_k \right| = \frac{(n - |B|)!}{(n-m)!}$$

Therefore,

$$|D_m| = \frac{(n)!}{(n-m)!} - \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \binom{n-i}{m-i} (m-i)!$$
$$= \binom{n}{m} m! - \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n-k}{m-k} (m-k)!$$

Thus,

$$|D_m| = \frac{(n)!}{(n-m)!} + \sum_{i=1}^{m} (-1)^i \binom{m}{i} \binom{n-i}{m-i} (m-i)! = \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \binom{n-i}{m-i} (m-i)!$$

Corollary. Assume that D_n is the number of derangement on n elements then

$$|D_n| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Proof. In previous result set $m = n$.

3 ACKNOWLEDGEMENTS

The author thanks the Research Council of the University of Garmsar for support.

REFERENCES

[1] M. Bóna, Combinatorics of Permutations, Chapman & Hall/CRC, 2004.
[2] Chenying Wang , Piotr Miska , István Mező, The r-derangement numbers, Discrete Mathematics, Vol. 340, 2017, 1681-1692.
[3] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, Addison Wesley, 1993.
[4] P. Miska, Arithmetic properties of the sequence of derangements, J. Number Theory 163 (2016) 114–145
[5]] M. Shattuck, Generalizations of Bell number formulas of Spivey and Mező, Filomat (in press)