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The graph burning problem for some families of graphs

Elham Roshanbin
Alzahra University, Department of Mathematics, Faculty of Mathematical Sciences
North Sheikh Bahaee St., Deh-e Vanak, Tehran, Iran
e.roshanbin@alzahra.ac.ir

Yong Gao
University of British Columbia-Okanagan, Computer Science Department
Irving K. Barber School of Arts and Sciences, Kelowna, BC, Canada V1V 1V5
yong.gao@ubc.ca

ABSTRACT

Here we consider the graph burning problem for some families of graphs including caterpillars and **AT**-free graphs, and we also study the asymptotic value of the burning number for the caterpillars in a random space.

KEYWORDS: graph burning, burning number, caterpillars, AT-free graphs

1 INTRODUCTION

Graph burning is a graph process that can be used as a model for the spread of social contagion and was introduced in 2014 by Anthony Bonato, Jeannette Janssen and Elham Roshanbin in [3]. Graph burning is defined on the node set of a simple finite graph G that can be seen as the underlying graph of a social network; it in fact represents a model for the spread of any sort of influence among the members of the corresponding network that are now the nodes of G. Initially we assume that all nodes of G are unburned (uninfluenced). We then at each discrete time step burn a node directly as a *fire source* and simultaneously the fire (influence) spread from the burning nodes of the previous stage to their unburned neighbours. Once a node is burned it remains burning until the end of this process that occurs when all nodes of G are burned. The *burning number* of G is the minimum number of steps that is needed for burning G and is denoted by $b(G)$.

It is clear that, the smaller the burning number of a graph is, the faster we could spread an influence among the users of a network in the above interpretation. If the burning process for a graph G ends in k steps by choosing the nodes $x_1, x_2, \dots, x_k$ as the fire sources respectively, then we call the sequence $(x_1, x_2, \dots, x_k)$ a *burning sequence* for G. For further results on the graph burning see [1, 2, 4, 9, 10]. The following facts about graph burning can be found in [4, 10].

Note that the only graph with burning number one is K_1. The following corollary from [4], is often used in the proof of the results on the burning number.

Corollary 1.1 ([4]). If $(x_1, x_2, \dots, x_k)$ is a sequence of nodes in a graph G, such that $N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \dots \cup N_0[x_k] = V(G)$, then $b(G) \leq k$.

The following theorem determines the burning number of paths.

Theorem 1.2 ([4]). For a path P_n we have that $b(P_n) = \lceil \sqrt{n} \rceil$.

From the proof of Theorem 1.2 in [4], we conclude that every burning sequence of a path P of order n corresponds to a partition for P into subpaths $Q_1, Q_2, \dots, Q_k$, in which the order of each Q_i is a number between one and $2i - 1$, and $k = \lceil \sqrt{n} \rceil$. Here in this paper, we call such a partition of P a *burning partition*.

A subgraph H of graph G is called an *isometric subgraph* of G if the distance between any pair of nodes u and v in H equals the distance between u and v in G. For example, any subtree of a tree T is an isometric subgraph of T. The following corollary is a generalization of Theorem 7 from [4] for disconnected graphs.

Corollary 1.3 ([10]). If G is a graph and H is an isometric subforest of G, then $b(H) \leq b(T)$.

The following lemma, provides tight bounds on the burning number of a connected graph in terms of its radius and diameter.

Lemma 1.4 ([4]). For every graph G with radius r and diameter d,

$$\lceil \sqrt{d+1} \rceil \leq b(G) \leq r+1.$$

2 BURNING NUMBER OF CATERPILLARS AND AT-FREE GRAPHS

In this section, we show that there are only two classes of caterpillars according to their burning number, and that determining in which class a given caterpillar belongs to, is **NP**-complete. We also consider the burning number of AT-free graphs. Here is the statement of the Graph Burning problem as decision problem.

Problem: Graph Burning
Instance: A simple graph G of order n and an integer $k \geq 2$.
Question: Is $b(G) \leq k$? In other words, does G contain a burning sequence $(x_1, x_2, \ldots, x_k)$?

In the PhD thesis [10] and the subsequent paper [2], it is proved by a creative reduction from a variant of the known 3-Partition problem (see [6]), called Distinct 3-Partition problem, that the Graph Burning problem is **NP**-complete even for trees of maximum degree three (In [8], it is shown that the Distinct 3-Partition problem is indeed strongly **NP**-complete (see [6]); which is needed in the proof.). Here is the statement of this problem.

Problem: Distinct 3-Partition
Instance: A finite set $X = \{a_1, a_2, \ldots, a_{3n}\}$ of positive distinct integers, and a positive integer B where $\sum_{i=1}^{3n} a_i = nB$, and $B/4 < a_i < B/2$, for $1 \leq i \leq 3n$.
Question: Is there any partition of X into n triples such that the elements in each triple add up to B?

A path P in a graph G is called a dominating path if for every node v in $V(G-P)$ there is a node in P that is adjacent to v. A caterpillar is a tree with a dominating path [5].

In this section, using exactly the same idea as in the above-mentioned proof from [10], we will show that the Graph Burning problem is **NP**-complete even for caterpillars (This result has been proved independently and similarly in [7]). Before that, we first explore the bounds on the burning number of caterpillars as follows (the upper-bound can also be found in [7]).

Theorem 2.1. Suppose that T is a caterpillar with a dominating path P of order n, and let $k = \lceil \sqrt{n} \rceil$. Then $k \leq b(T) \leq k+1$. Moreover, if $n = k^2$, then $b(T) = k$ if and only if there is a burning partition for P into subpaths $Q_1, \ldots, Q_k$ such that there is no node in $T - P$ joined to the end points of Q_i's. If $n < k^2$, then $b(T) = k$ if and only if one of the following conditions holds:

(i) There is a burning partition for P into subpaths $Q_1, \ldots, Q_k$ such that there is no node in $T - P$ joined to the end points of Q_i's.

(ii) There is a partition for P into subpaths $Q_1, \ldots, Q_{k-1}$, in which each Q_i is a path of order $1 \leq l_i \leq 2(k-i)+1$ and there is at most one node in $T - P$ that is joined to an end point of one of the Q_i's.

(iii) There is a partition for P into subpaths $Q_1, \ldots, Q_{k-1}$, in which each Q_i is a path of order $1 \leq l_i \leq 2(k-i)+1$, for $1 \leq i \leq k-2$, and Q_{k-1} is of order one, and there is at most one node in $T - P$ that is joined to an end point of one of the Q_i's.

(iv) There is a partition for P into subpaths $Q_1, \ldots, Q_{k-2}$, in which each Q_i is a path of order $1 \leq l_i \leq 2(k-i)+1$ and there are at most only two nodes in $T - P$ that are joined to the end points of some of the Q_i's.

Proof. There are two possibilities for n: either $n = k^2$ or $n < k^2$. Note that P is an isometric subpath of T and therefore by Corollary 1.2 and Corollary 1.3, $b(T) \geq b(P) = k$. On the other hand, if $(x_1, \dots, x_k)$ is a burning sequence for P, then $\{N_{k+1-i}[x_i\ ]\}_{i=1}^{k}$ forms a covering for the node set of T. Thus, we conclude that $b(T) \leq k + 1$. Also, note that if $(x_1, \dots, x_k)$ is a burning sequence for T, then (each x_i must be either in P or in $T - P$) clearly $Q_i = N_{k-i}[x_i] \cap P$ is a subpath of T of order at most $2(k - i) + 1$.

First, assume that $n = k^2$ and $(x_1, \dots, x_k)$ is an optimum burning sequence for T. Since $n = k^2$, we conclude that each $Q_i = N_{k-i}[x_i] \cap P$ must be a path of order $2(k - i) + 1$, and $\{Q_i\}_{i=1}^{k}$ forms a partition for P. Moreover, $x_i \in P$ for $1 \leq i \leq k$, and there must not be any node x outside of P that is joined to an end point of a Q_i; otherwise, it implies that x will not be burned by the end of the k-th step, which is a contradiction.

Conversely, suppose that there is a burning partition for P into subpaths $Q_1, \dots, Q_k$ such that there is no node in $T - P$ joined to the end points of Q_i's. Let x_i be a central node of Q_i, for $1 \leq i \leq k$. Then we can easily see that $(x_1, \dots, x_k)$ is a burning sequence for T, and therefore, by the argument in the first paragraph of the proof, we conclude that $b(T) = k$.

Now assume that $n < k^2$, and $(x_1, \dots, x_k)$ is an optimum burning sequence for T. For $1 \leq i \leq k$, let $Q_i = \{u \in N_{k-i}[x_i] : i = min\{j : u \in N_{k-j}[x_j]\}\}$. We can easily see that Q_i's form a partition for P. There are two possibilities for x_k: either $x_k \in V(P)$ or $x_k \notin V(P)$. Similarly, there are two possibilities for x_{k-1}: either $x_{k-1} \in P$ or not. Hence, we have to consider four different cases as follows:

(i) If $x_{k-1}, x_k \in P$, then let $Q_i = N_{k-i}\ [x_i] \cap P$, for $1 \leq i \leq k$. We can easily see that $\{Q_i\}_{i=1}^{k}$ forms a burning partition for P into subpaths such that there is no node x in $T - P$ that is joined to the end points of Q_i's. Otherwise, x cannot be burned by the end of the k-th step. Conversely, in such a case, assume that there is a burning partition for P into subpaths $Q_1, \dots, Q_k$ such that there is no node in $T - P$ joined to the end points of Q_i's. For each $1 \leq i \leq k$, let x_i be a central node in Q_i. Then $\{N_{k-i}[x_i\ ]\}_{i=1}^{k}$ forms a covering for $V(T)$. Hence, by Corollary 1.1, we conclude that $b(T) \leq k$.

(ii) If $x_{k-1} \in P$ and $x_k \notin P$, then let $Q_i = N_{k-i}[x_i] \cap P$, for $1 \leq i \leq k - 1$. We can easily see that $\{Q_i\}_{i=1}^{k-1}$ forms a partition for P into subpaths $Q_1, \dots, Q_{k-1}$, in which each Q_i is a path of order $1 \leq l_i \leq 2(k - i) - 1$ and there is at most one node in $T - P$, that can be only x_k, that is joined to an end point of one of the Q_i's (otherwise, there must be some node that cannot be burned by the end of the k-th step, which is a contradiction). Conversely, in such a case, assume that there is a partition for P into subpaths $Q_1, \dots, Q_{k-1}$ in which each Q_i is of order $1 \leq l_i \leq 2(k - i) - 1$, and there is at most one node in $T - P$ joined to an end point of one of the Q_i's. Let x_i be a central node in Q_i, for $1 \leq i \leq k - 1$. If there is no node in $T - P$ that is adjacent to an end point of one of the Q_i's then we take x_k to be an arbitrary node in $T - \cup_{i=1}^{k-1} N_{k-i-1}[x_i]$ (We know that such a node does exist since $b(T) \geq k$); otherwise, let x_k to be the node that is adjacent to one of the end points of the Q_i's (as by assumption there can be at most one node like that). Then $\{N_{k-i}[x_i\ ]\}_{i=1}^{k}$ forms a covering for $V(T)$. Hence, by Corollary 1.1, we conclude that $b(T) \leq k$.

(iii) If $x_{k-1} \notin P$ and $x_k \in P$, then let $Q_i = N_{k-i}[x_i\ ] \cap P$, for $1 \leq i \leq k - 2$, and $Q_{k-1} = x_k$. We can easily see that $\{Q_i\}_{i=1}^{k-1}$ forms a partition for P into subpaths $Q_1, \dots, Q_{k-1}$, in which each Q_i is a path of order $1 \leq l_i \leq 2(k - i) - 1$, for $1 \leq i \leq k - 2$. Moreover, the path Q_{k-1} is of order one, and there is at most one node in $T - P$, that can be only x_{k-1}, that is joined to an end point of one of the Q_i's. Conversely, in such a case, assume that there is a desired partition for P into subpaths $Q_1, \dots, Q_{k-1}$. Let x_i be a central node in Q_i, for $1 \leq i \leq k - 2$. Now, if there is no node in $T - P$ that is adjacent to an end point of a Q_i, then take x_{k-1} to be an arbitrary node in $T - \cup_{i=1}^{k-2}\ N_{k-2-i}\ [x_i]$ (We know that such a node does exist since $b(T) \geq k$); otherwise, let x_{k-1} to be the node that is adjacent to one of the end points of the Q_i's (as by assumption there can be at most one node like that). Moreover, let x_k be the only node in Q_{k-1}. Then $\{N_{k-i}[x_i\ ]\}_{i=1}^{k}$ forms a covering for $V(T)$. Hence, by Corollary 1.1, we conclude that $b(T) \leq k$.

(iv) If $x_{k-1} \notin P$ and $x_k \notin P$, then let $Q_i = N_{k-i}[x_i\ ] \cap P$, for $1 \leq i \leq k - 2$. We can easily see that $\{Q_i\}_{i=1}^{k-2}$ forms a partition for P into subpaths $Q_1, \dots, Q_{k-2}$, in which each Q_i is a path of order $1 \leq l_i \leq$

$2(k-i)-1$, for $1 \leq i \leq k-2$. Moreover, there are at most two nodes in $T-P$, that can be only x_{k-1} and x_k, that are adjacent to the end points of one of the Q_i's. Conversely, in such a case, assume that there is a desired partition for P into subpaths $Q_1, \ldots, Q_{k-2}$. Let x_i be a central node in Q_i, for $1 \leq i \leq k-2$. Now, if there is no node in $T-P$ that is adjacent to an end point of a Q_i, then take x_{k-1} to be an arbitrary node in $T - \cup_{i=1}^{k-2} N_{k-2-i}[x_i]$ (We know that such a node does exist since $b(T) \geq k$); otherwise, let x_{k-1} to be a node that is adjacent to one of the end points of the Q_i's. Similarly, if there is no node $x \neq x_{k-1}$ in $T-P$ that is adjacent to an end point of a Q_i, then take x_k to be an arbitrary node in $T - \cup_{i=1}^{k-1} N_{k-1-i}[x_i]$ (We know that such a node does exist since $b(T) \geq k$); otherwise, let x_k to be a node $x \neq x_{k-1}$ that is adjacent to one of the end points of the Q_i's (as by assumption there can be at most one node like that). Then $\{N_{k-i}[x_i]\}_{i=1}^{k}$ forms a covering for $V(T)$. Hence, by Corollary 1.1, we conclude that $b(T) \leq k$.

A set of three disjoint nodes u_1, u_2, and u_3 in a graph G is called an *asteroidal triple* or **AT** of G if for every pair of u_i's, there is a path connecting these two nodes that does not intersect with the neighbourhood of the third node. A graph G is ***AT**-free* if it contains no asteroidal triple of nodes. A pair of nodes u and v in a graph G is called a *dominating pair* of G if every path in G that connects u and v is a dominating path for G. The following theorem is known about the AT-free graphs.

Theorem 2.2 ([5], Theorem 7.2.8). Every connected **AT**-free graph has a dominating pair.

We also know the following fact about the interval graphs.

Theorem 2.3 ([5], Theorem 7.2.6). A graph G is an interval graph if and only if it is chordal and **AT**-free.

We can conclude the following result from the last three theorems and Lemma 1.4.

Corollary 2.4. If G is a connected **AT**-free graph or a connected interval graph of diameter d, then

$$\lceil\sqrt{d+1}\rceil \leq b(G) \leq \lceil\sqrt{d+1}\rceil + 1$$

Let P_n be the dominating path in a caterpillar T such that $T-P_n$ consists of t pendant nodes or legs. By Theorem 2.1, we already know that $\lceil\sqrt{n}\rceil \leq b(T) \leq \lceil\sqrt{n}\rceil + 1$. We can easily see that $b(T) = \lceil\sqrt{n}\rceil$ if and only if there is a partition of P_n into subpaths of orders $1, 3, \ldots, 2k-1$, such that there is no leg attached to the end points of any of these subpaths. Using this observation, in Theorem 2.5, we show that it is hard to determine the exact class of T according to its burning number (This result also has been proved in [7]).

Theorem 2.5. The Graph Burning problem is **NP-**complete even for caterpillars.

Proof. Clearly, the Graph Burning problem is in **NP**. Suppose that we have an instance of the Distinct 3-Partition problem; that is, we are given a non-empty finite set $X = \{a_1, a_2, \ldots, a_{3n}\}$ of distinct positive integers, and a positive integer B such that $\sum_{i=1}^{3n} a_i = nB$, and $B/4 < a_i < B/2$, for $1 \leq i \leq 3n$. Since the Distinct 3-Partition problem is **NP**-complete in the strong sense, without loss of generality we can assume that B is bounded above by a polynomial in the length of the input.

Assume that the maximum of the set X is m which is by assumption bounded above by $B/2$. Let $Y = \{2a_i - 1 : a_i \in X\}$. Hence, $Y \subseteq O_m$, and $2nB - 3n = \sum_{i=1}^{3n}(2a_i - 1)$ is the sum of the numbers in Y. Let $Z = O_m \backslash Y$. Note that $1 \leq |Y| \leq m$, and consequently, $|Z| \leq m - 1$. Let $|Z| = k$, for some $k \leq m-1$. For $1 \leq i \leq k$, let Q_i' be a path of order l_i, where l_i is the i-th largest number in Z. For $1 \leq i \leq m+1$, we define Q_i'' to be a path of order $2(2m+1-i)+1$. We also take Q_i to be a path of order $2B-3$, for $1 \leq i \leq n$. We construct a path P of order $\sum_{i=1}^{2m+1}(2i-1) = (2m+1)^2$, that is obtained by adding an edge between the end points of two successive paths in the following order:

$$Q_1, Q_1'', Q_2, Q_2'', \ldots, Q_n, Q_n'', Q_1', Q_{n+1}'', Q_2', Q_{n+1}'', \ldots, Q_k', Q_{n+1}'', Q_{n+k+1}'', \ldots, Q_{m+1}''.$$

Assume that $\{x_1, \ldots, x_s\}$ is the set of end points of the paths Q_i''s and Q_j'' 's. Let $G(P)$ be the graph that is obtained by joining every node $u \in V(P) \setminus (\left(\cup_{i=1}^{n} V(Q_i)\right) \cup \{x_1, \ldots, x_l\})$ to a new node. Clearly, $G(P)$ is a caterpillar graph. We can easily see that there is a partition of X into triples such that the elements in each triple add up to B if and only if we can decompose the paths $Q_1, Q_2, \dots, Q_n$ into subpaths of orders $2a_i - 1 \in Y$.

Now, assume that there is a partition of X into triples such that the elements in each triple add up to B. Equivalently, we have a partition for the paths $Q_1, Q_2, \dots, Q_n$ in terms of subpaths $\{P_l : l \in Y\}$. Since $O_m = Y \cup Z$, we conclude that there is a partition for the subgraph $(\bigcup_{i=1}^{n} Q_i) \cup \big(\bigcup_{i=1}^{k} Q_i'\big) \cup (\bigcup_{i=1}^{m+1} Q_i'')$ in terms of the subpaths $\{P_l : l \in O_m\}$. Now, for $m+2 \le i \le 2m+1$, let x_i be the centre of a path P_l in such a partition, where $l = 2(2m+2-i) - 1 \in O_m (= Y \cup Z)$. For $1 \le i \le m+1$, let x_i be the centre of Q_i''. Thus, we have that

$$V(G(P)) = \bigcup_{i=1}^{2m+1} N_{2m+1-i}[x_i] .$$

Consequently, by Corollary 1.1, we conclude that $(x_1, \dots, x_{2m+1})$ forms a burning sequence of length $2m+1$ for $G(P)$. Therefore, $b\big(G(P)\big) \le 2m+1$.

Conversely, suppose that $b\big(G(P)\big) \le 2m+1$. Note that the path P of order $(2m+1)^2$ is a subtree of $G(P)$. Therefore, by Theorem 1.2 and Corollary 1.3, we have that

$$b\big(G(P)\big) \ge b(P) = 2m+1.$$

Thus, we conclude that $b(G(P)) = 2m+1$.

Assume that $(x_1, \dots, x_{2m+1})$ is an optimum burning sequence for $G(P)$. We first claim that each x_i must be in P. Note that every x_i is either in P, or joined to a node $u \in V(P)$. On the other hand, every node in P must receive the fire from one of the x_i's. Hence, for $1 \le i \le 2m+1$, $N_{2m+1-i}[x_i] \cap P$ must be a path of order at most $2(2m+1-i)+1$. If for some $1 \le i \le 2m+1$, the node x_i is not in P, then $N_{2m+1-i}[x_i] \cap P$ is a path of order less than $2(2m+1-i)+1$. Therefore, the total sum of the orders of the subpaths $\{N_{2m+1-i}[x_i] \cap P\}_{i=1}^{2m+1}$ will be less than $(2m+1)^2 = |V(P)|$, which is a contradiction. Thus, every x_i must be selected from P.

Now, we claim that for $1 \le i \le m+1$, the fire source x_i must be the middle node of Q_i''. We prove this by strong induction on i. Note that Q_1'' is the largest path in constructing $G(P)$ in which there is a node attached to all of its non-end point nodes. Hence, we can easily see that if we chose x_1 to be any node in P rather than the centre of Q_1'', then there will be a node joined to at least one of the end points of $N_{2m+1-1}[x_1] \cap P$, which leads to a contradiction. Thus, x_1 must be the centre of Q_1''. Suppose that for $1 \le i \le m$ and for every $1 \le j \le i$, x_j is the centre of Q_j''. Since Q_{i+1}'' is the $(i+1)$-th largest path in constructing $G(P)$ such that all of its non-end point nodes are adjacent to a node in $G(P) - P$, and by induction hypothesis, we conclude that x_{i+1} must be the centre of Q_{i+1}''. Therefore, the claim is proved by induction.

The above argument implies that the nodes in $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ must be burned by receiving the fire started at $x_{m+2}, x_{m+3}, \dots, x_{2m+1}$ (the last m sources of fire). Since $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ is a disjoint union of paths, then we derive that for $m+2 \le i \le 2m+1$, $N_{2m+1-i}[x_i] \cap (G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i''))$ is a path of order at most $2(2m+1-i)+1$ $(\le 2m-1)$. On the other hand, the path-forest $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ is of order $\sum_{i=1}^{m}(2i-1) = m^2$. Thus, we conclude that for $m+2 \le i \le 2m+1$, $N_{2m+1-i}[x_i] \cap (G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i''))$ is a path of order $2(2m+1-i)+1$; since otherwise, we cannot burn all the nodes in $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ in m steps, which is a contradiction. Therefore, there must be a partition of $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ (induced by the burning sequence $(x_{m+2}, x_{m+3}, \dots, x_{2m+1})$) for $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i''))$ into subpaths $\{P_l : l \in O_m\}$.

Now, considering the partition described in the previous paragraph, we claim that there is a partition of $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ into subpaths of orders in O_m in which the paths $Q_1, Q_2, \dots, Q_n$ are decomposed into paths of orders in Y, and each path Q_i' is covered by itself. Note that by definition, for $1 \le i \le k$, each path Q_i' is a component of $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$. Hence, it suffices to prove that there is a partition of $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ into subpaths of orders in O_m such that each Q_i' is covered by itself. Assume that in a partition of $G(P) \setminus (\bigcup_{i=1}^{m+1} Q_i'')$ into subpaths of orders in O_m, there is a path Q_i' of order $l \in O_m \setminus Y (= Z)$ that is partitioned by a union of paths of orders in O_m rather than by P_l itself. We know that P_l must have covered some part of a path Q'_j, where $j \neq i$, or must be used in partitioning $Q_1, Q_2, \dots, Q_n$. Hence, we can easily modify the partition by switching the place of P_l and those paths that have covered Q'_i (as they have equal lengths). Therefore, we have decreased the number of such displaced paths in our partition for $G(P) \setminus (\bigcup_{i=1}^{m+1} Q''_i)$. Since the number of Q'_i's, where $1 \leq i \leq k$, is finite, we will end up after finite number of switching in a partition for $G(P) \setminus (\bigcup_{i=1}^{m+1} Q''_i)$ in which every Q'_i, $1 \leq i \leq k$, is covered by itself.

Finally, since each Q_i is of order $2B-3$, there must be a partition of Y into triples such that the elements in each triple add up to $2B-3$. Equivalently, there must be a partition of X into triples such that the elements in each triple add up to B. Since $G(P)$ is a caterpillar, we have a polynomial time reduction from the Distinct 3-Partition problem to the Graph Burning problem for caterpillars.

Since every caterpillar is an interval graph and also it is an **AT**-free graph, we have the following immediate corollary.

Corollary 2.6. The Graph Burning problem is **NP-**complete even for interval graphs and for **AT**-free graphs.

3 BURNING NUMBER OF CATERPILLARS IN A RANDOM SPACE

In this section, we obtain some results on the asymptotic value of the burning number of the caterpillars in a random space that we define as follows.

We can look at the set of all caterpillars with dominating path P_n and t legs as a random space, denoted by $\mathcal{R}$, that its elements are generated in the following way: In a random experiment, we choose t nodes in P_n uniformly at random (the nodes can be the same; in other words, we choose them with replacement), and we attach to each node a leg. We consider the asymptotic value of the burning number of a graph in $\mathcal{R}$ and we prove the following theorem.

Theorem 2.6. Let T be a caterpillar in $\mathcal{R}$ with dominating path P_n and t legs. Then a.a.s.,

$$b(T) = \begin{cases} b(P_n), & if\ t = o(n) \\ b(P_n) + 1, & if\ t = \Omega(n) \end{cases}$$

Proof. Remember that $b(T) = \lceil \sqrt{n} \rceil$ if and only if there is a partition of P_n into subpaths of orders $1, 3, \dots, 2k-1$, such that there is no leg attached to the end points of any of these subpaths. We call such a partition, a *good partition*. Let Y_i be the indicator random variable for the event A_i that is defined as follows: A_i occurs if none of the end points of the subpaths in a partition Π_i (one of the many possible partitions) are chosen in a random experiment that creates a member of $\mathcal{R}$. We may think of this experiment as *playing darts*. We have this path P_n as the dart board, and the partition segments are like the rings on the board. Now imagine we aim to hit the end points of the segments. When we throw enough number of darts, at least some of them should hit the end points of the segments.

We can easily see that

$$\mathbb{P}(Y_i = 1) = \left(1 - \frac{(2k-1)}{n}\right)^t \sim e^{-\frac{2t}{\sqrt{n}}}.$$

As an immediate observation, note that if $t << \sqrt{n}$, then $\mathbb{P}(Y_i = 1)$ goes to 1 as n goes to infinity. Thus, in this case, a.a.s., none of the end points of the segments in the partition Π_i of P_n will have an attached leg. Hence, the burning number of a caterpillar $T \in \mathcal{R}$ with t legs, where $t << \sqrt{n}$, a.a.s., equals the burning number of P_n, i.e., $\lceil \sqrt{n} \rceil$.

First suppose that t is at least linear; that is, $t \geq cn$ for some constant c. Let $v_1, \dots, v_n$ be the nodes of the path P_n according to the order of their appearance. Note that v_1 and v_n appear as end points in all of the possible partitions. Hence, to avoid hitting end points in any partition of P_n in the mentioned experiment, we always have to avoid hitting v_1 and v_n. Let A be the event where we avoid hitting v_1 and v_n; and let B

be the event where we avoid hitting any of the end points in any of the partitions. We can easily see that $B \subseteq A$. Thus, we have that

$$\mathbb{P}(Y_i \geq 1) = \mathbb{P}(B) \leq \mathbb{P}(A) = \left(1 - \frac{2}{n}\right)^t \leq\ e^{-\frac{2t}{n}} \leq e^{-\frac{2c}{n}}.$$

Therefore,

$$\mathbb{P}(Y_i = 0) \geq 1 - e^{-\frac{2c}{n}} > 0.$$

It implies that if $t = \Omega(n)$, then there is a partition where we avoid hitting any of the end points.

Now let t be sublinear; that is, $t = o(n)$. Let assume that we have thrown t darts randomly, and then starting from v_1, we try to construct a *good partition* for P_n and calculate the chance of having such a partition as follows at the same time. Note that the orders of the paths in the partition are coming from the set $O_k = \{1, 3, \ldots, 2k - 1\}$. Moreover, note that the end points of the segments of a partition come in pairs (except the last end point). We call such a pair a *boundary* point. It is easy to see that in any partition of P_n into paths of orders $\{1, 3, \ldots, 2k - 1\}$, the distance between every two successive boundary points is at least two. Thus, boundary points are disjoint (two successive endpoints that form a boundary point, are disjoint from those that form another boundary point).

Note that v_1 appears as the subpath of order 1 in a good partition in this random experiment, only if v_2 is the end point of another segment; i.e., the pair (v_1, v_2) forms a boundary point for such a partition. We can easily see that, the pair (v_1, v_2) appears in a good partition with probability

$$\left(1 - \frac{2}{n}\right)^t \geq 1 - \frac{2t}{n}.$$

Therefore, the probability that the pair (v_1, v_2) does not appear in a good partition is at most $\frac{2t}{n}$.

Now for $i = 2$, we consider the possibility of choosing a pair of nodes in P_n after v_2, that can be taken as the i-th boundary point in a good partition. For $i = 2$, we have to see if we can find a couple of successive nodes in P_n after v_2 such that none of them are hit by any dart. For this purpose, we have to check all of the possible nodes v_{2+2i}, for $1 \leq i \leq k - 1$ (the distance between v_2 and v_{2i+2} is an even number) to see if v_{2+2i} appears in a boundary point or not, and if it is hit by a dart or not. Since the boundary points are disjoint, all of these events are independent. Therefore, the probability that we cannot find such a desired boundary point as explained is at most $\left(\frac{2t}{n}\right)^{k-1}$.

For $2 \leq i \leq k$, with a similar discussion, we can see that the probability of not being able to choose a pair of successive nodes that form the i-th boundary point in a good partition of P_n is at most $\left(\frac{2t}{n}\right)^{k-i+1}$. Thus, the probability of not being able to construct a good partition starting from v_1 as described above is at most

$$\frac{2t}{n} + \sum_{i=1}^{k-1} \left(\frac{2t}{n}\right)^i = O(\frac{2t}{n}),$$

which goes to zero as n goes to infinity (since by assumption $t = o(n)$). Therefore, for $t = o(n)$, a.a.s., there is a good partition and the proof follows.

REFERENCES

[1] S. Bessy, A. Bonato, J. Janssen, D. Rautenbach, E. Roshanbin, "Bounds on the Burning Number", Discrete Applied Mathematics 235 (2018) 16-22.

[2] S. Bessy, A. Bonato, J. Janssen, D. Rautenbach, E. Roshanbin, "Burning a Graph Is Hard", Discrete Applied Mathematics 232 (2017) 73-87.

[3] A. Bonato, J. Janssen, E. Roshanbin, "Burning a Graph as a Model of Social Contagion", Lecture Notes in Computer Science 8882, (2014), 13-22.

[4] A. Bonato, J. Janssen, E. Roshanbin, "How to burn a graph", Internet Mathematics 1-2 (2016) 85-100.

[5] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM, 1999.

[6] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of **NP**-Completeness, W.H. Freeman, 1979.

[7] M. Hiller, E. Triesch, A. M.C.A. Koster, "On the Burning Number of p-Caterpillars", 2019, arXiv: 1912.10897.

[8] H. Hulett, T.G. Will, G.J. Woeginger, "Multigraph realizations of degree sequences: Maximization is easy, minimization is hard", Operations Research Letters **36** (2008) 594-596.

[9] D. Mitsche, P. Pralat, E. Roshanbin, "Burning graphs-a probabilistic perspective", Graphs and Combinatorics 33(2) (2017), 449-471.

[10] E. Roshanbin, Burning a graph as a model of social contagion, PhD Thesis, Dalhousie University, 2016.