Bounds on Partition Dimension of Peterson Graphs

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ABSTRACT

The distance of a connected, simple graph $\mathbb{P}$ is denoted by $d(\eta_1, \eta_2)$, which is the length of a shortest path between the vertices $\eta_1, \eta_2 \in V(\mathbb{P})$, where $V(\mathbb{P})$ is the vertex set of $\mathbb{P}$. The l-ordered partition of $V(\mathbb{P})$ is $\theta = \{\theta_1, \theta_2, \ldots, \theta_l\}$. A vertex $\eta \in V(\mathbb{P})$, and $r(\eta|\theta) = \{d(\eta, \theta_1), d(\eta, \theta_2), \ldots, d(\eta, \theta_l)\}$ be a l-tuple distances, where $r(\eta|\theta)$ is the representation of a vertex η with respect to set θ. If $r(\eta|\theta)$ of η is unique, for every pair of vertices, then θ is the resolving partition set of $V(\mathbb{P})$. The minimum number l in the resolving partition set θ is known as partition dimension ($pd(\mathbb{P})$). In this paper, we studied the generalized families of Peterson graph, $P_{\lambda,\lambda-1}$ and proved that these families have bounded partition dimension.

KEYWORDS: Generalized Peterson graph, partition dimension, partition resolving set, sharp bounds of partition dimension

1 INTRODUCTION

In 1975 the idea delivered by Slater had a background in networking, usually referred to as locating set or beacons set. The entire network or a graph is controlled by specifically chosen vertices from the vertex set in this concept. These vertices have to choose with a specific condition that each vertex of a graph has a unique position in terms of representations, we refer to the Definition 1.2 for this concept. Later Melter and Harary rename this concept as resolving set [10]. In the graph's theoretical study, this concept is called a metric basis or basis set of a graph. The count of vertices in a resolving set or metric basis is referred as the metric dimension of a graph [18]. Instead of choosing particular nodes into a subset with the defined condition, it is possible to arrange the entire vertex set into subsets keeping the defined condition of $r(\eta|\theta)$, which is actually came from the idea of the unique position of each vertex in a graph. This concept is called as the partition resolving set, and the least number of subsets is called the partition dimension, introduced by Chartrand et al. in 2000 [5]. To better understand this concept, we refer to the mathematical Definition 1.3 and 1.4 and for the latest ideas related to this concept, see [9].

Representing a graph with each of its vertex has unique position is falling in different real-world applications, such as for the strategies, coding, and decoding of mastermind games brief in [8], the popular relation which is named as Djokovic-Winkler linked to this concept [4], the piloting or the guidance of a robot also associated with this unique idea [14], the procedure of verifying and discovering a network related to this concept [3]. There are many applications to explore those, we refer to see [10]. Finding of a resolving set is NP-hard problem [11] and the partition resolving set is the generalization of resolving set it also falls in the category of NP-hard [5].

The abstraction of resolving partition set and partition dimension extensively occurred in the literature. For example, the graph with partition dimension $|V|-3$ discussed [2], the graph obtained by few graph operations and its corresponding partition dimension studied in [19], bounds on the partition dimension for convex polytopes in [6, 7], bounds of partition on the circulant and multipartite discussed in [15], on the bounded partition dimension of the Cartesian product of graphs are studied in [21], [1] gave bounds for the subdivision of different graphs. For the resolving set and metric dimension of Peterson and generalized Peterson graph, we refer to the articles [12, 16]. For more recent literature and results, we refer to see [6, 7, 17].

Following are basic mathematical definitions of the concepts used in this research work and useful Theorems.

Definition 1.1 *Suppose P be an undirected, simple graph with the set of vertices named as $V(P)$ and edge set $E(P)$, the distance which also known as geodesics, between $\eta_1, \eta_2 \in V(P)$ two vertices is the count of minimum edges between $\eta_1 - \eta_2$ path. It is denoted by $d(\eta_1, \eta_2)$.*

Definition 1.2 *Suppose an ordered set of vertices from $V(P)$ labeled as $R = \{\eta_1, \eta_2, \dots, \eta_s\}$ and $\alpha \in V(P)$. The representations $r(\eta|R)$ of η-vertex with respect to an ordered subset R is the $s-$tuple distances $(d(\eta,\eta_1), d(\eta,\eta_2), \dots, d(\eta,\eta_s))$. If each vertex from $V(P)$ have unique representations according to R, then R is called a resolving set of graph P, and minimum count of the elements in R is called the metric dimension of graph P and it is represented by $dim(P)$.*

Definition 1.3 *Let θ is the l-ordered partition set and $r(\eta|\theta) = \{d(\eta,\theta_1), d(\eta,\theta_2), \dots, d(\eta,\theta_l)\}$, is the l-tuple distance representations of a vertex η with respect to θ. If the representations of η with respect to θ are unique, then θ is the partition resolving set of the vertex set of a graph P.*

Definition 1.4 *The minimum count of subsets in the partition resolving set of $V(P)$ is defined as the partition dimension $(pd(P))$ of P.*

Theorem 1.5 *[5] Let θ be a partition resolving set of $V(P)$ and $\eta_1, \eta_2 \in V(P)$. If $d(\eta_1,\eta) = d(\eta_2,\eta)$ for all vertices $\eta \in V(P)\backslash(\eta_1,\eta_2)$, then η_1, η_2 belongs to different subsets of θ.*

1.1 Generalized Peterson Graphs: $P_{\lambda,\lambda-1}$

The generalized Petersen graph $P_{\lambda,\lambda-1}$ is a graph with $\lambda \geq 2$ having vertex set $\eta \cup \zeta$ where $\{\eta_\xi, \zeta_\xi\colon \xi = 1,2,\dots,2\lambda\}$. For our purpose, we call the vertices $\eta_1, \eta_2, \dots, \eta_{2\lambda}$ outer vertices and $\zeta_1, \zeta_2, \dots, \zeta_{2\lambda}$ inner vertices.

Theorem 2.1 *Let $P_{\lambda,\lambda-1}$ be generalized Petersen graphs with $\lambda \geq 2$ and $\lambda \equiv 0(mod\ 4)$. Then $pd(P_{\lambda,\lambda-1}) \leq 5$.*

Proof. For $\lambda = 4$, it is easy to see that $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_1\}$, $\theta_2 = \{\zeta_2\}$, $\theta_3 = \{\zeta_3\}$, $\theta_4 = \{\eta_4\}$, $\theta_5 = V(P_{\lambda,\lambda-1})\backslash\{\zeta_1,\zeta_2,\zeta_3,\eta_4\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$.

For $\lambda > 4$, $\lambda \equiv 0(mod\ 4)$ and for the resolving partition set $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_1\}$, $\theta_2 = \{\eta_3\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = \{\zeta_{\lambda+3}\}$, $\theta_5 = V(P_{\lambda,\lambda-1})\backslash\{\zeta_1,\zeta_{\lambda+3},\eta_3,\eta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$.

For $\lambda = 8$, the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are in the Tables 1 and 2.

Table 1: Representations of the outer vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5
η_1	1	2	4	3	0	η_2	2	1	3	2	0
η_4	4	1	5	2	0	η_5	5	2	4	3	0
η_6	4	3	3	4	0	η_7	3	4	2	5	0
η_8	2	5	1	4	0	η_{10}	2	3	1	2	0
η_{11}	3	4	2	1	0	η_{12}	4	3	3	2	0
η_{13}	5	4	4	3	0	η_{14}	4	5	5	4	0
η_{15}	3	4	4	5	0	η_{16}	3	2	2	1	0

Table 2: Representations of inner vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5
ζ_2	2	3	3	4	0	ζ_3	2	1	3	4	0
ζ_4	5	2	4	1	0	ζ_5	4	3	5	4	0
ζ_6	3	4	4	3	0	ζ_7	4	5	3	4	0
ζ_8	1	4	2	5	0	ζ_9	4	3	1	2	0
ζ_{10}	1	2	2	3	0	ζ_{12}	3	2	4	3	0
ζ_{13}	4	3	5	2	0	ζ_{14}	5	4	4	5	0
ζ_{15}	2	5	3	4	0	ζ_{16}	3	4	2	3	0

It is clear, one can be see easily that all the vertices which are $V(P_{\lambda,\lambda-1})$ for $\lambda = 8$ have unique representations with respect to θ.

Now for $\lambda > 8$ the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are: $r(\eta_{\varepsilon+1}: 0 \le \varepsilon \le 1|\theta) = (\varepsilon + 1,\ 2 - \varepsilon,\ 4 - \varepsilon,\ 3 - \varepsilon,\ 0)$, $r(\eta_{\varepsilon+1+2\psi}: 0 \le \varepsilon \le 2|\theta) = (2\psi - \varepsilon + 1,\ 2\psi + \varepsilon - 2,\ 2\psi - \varepsilon,\ 2\psi + j - 1,\ 0)$, $r(\eta_{\varepsilon+2+\lambda}: 0 \le \varepsilon \le 1|\theta) = (\varepsilon + 2,\ 3 + \varepsilon,\ \varepsilon + 1,\ 2 - \varepsilon,\ 0)$, $r(\eta_{2\psi+2+\lambda}) = (2\psi,\ 2\psi + 1,\ 2\psi + 1,\ 2\psi,\ 0)$, $r(\zeta_{\varepsilon+1+\lambda}: 0 \le \varepsilon \le 1|\theta) = (4 - 3\varepsilon,\ 3 - \varepsilon,\ \varepsilon + 1,\ \varepsilon + 2,\ 0)$, $r(\zeta_{\varepsilon+1+2\psi+\lambda}: 0 \le \varepsilon \le 1|\theta) = (2\psi + \varepsilon,\ 2\psi + \varepsilon - 1,\ 2\psi - \varepsilon + 1,\ 2\psi + 3\varepsilon - 2,\ 0)$, $r(\zeta_{2\psi+3+\lambda}|\theta) = (2\psi - 2,\ 2\psi + 1,\ 2\psi - 1,\ 2\psi,\ 0)$, and remaining in Tables 3 and 4, where $\psi = \frac{\lambda}{4}$.

Table 3: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	
$\eta_{\varepsilon+4}$	$\varepsilon+4$	$\varepsilon+1$	$\varepsilon+5$	$\varepsilon+2$	0	$0 \le \varepsilon \le 2\psi-4$
$\eta_{\lambda-\varepsilon}$	$\varepsilon+2$	$\varepsilon+5$	$\varepsilon+1$	$\varepsilon+4$	0	$0 \le \varepsilon \le 2\psi-4$
$\eta_{\lambda+\varepsilon+4}$	$\varepsilon+4$	$\varepsilon+3$	$\varepsilon+3$	$\varepsilon+2$	0	$0 \le \varepsilon \le 2\psi-3$
$\eta_{2\lambda-\varepsilon}$	$\varepsilon+2$	$\varepsilon+3$	$\varepsilon+3$	$\varepsilon+4$	0	$0 \le \varepsilon \le 2\psi-3$
$\zeta_{2\varepsilon+4}$	$2\varepsilon+5$	$2\varepsilon+2$	$2\varepsilon+4$	$2\varepsilon+1$	0	$0 \le \varepsilon \le \psi-2$
$\zeta_{2\varepsilon+5}$	$2\varepsilon+4$	$2\varepsilon+3$	$2\varepsilon+5$	$2\varepsilon+4$	0	$0 \le \varepsilon \le \psi-2$
$\zeta_{\lambda-2\varepsilon}$	$2\varepsilon+1$	$2\varepsilon+4$	$2\varepsilon+2$	$2\varepsilon+5$	0	$0 \le \varepsilon \le \psi-2$
$\zeta_{\lambda-2\varepsilon-1}$	$2\varepsilon+4$	$2\varepsilon+5$	$2\varepsilon+3$	$2\varepsilon+4$	0	$0 \le \varepsilon \le \psi-2$
$\zeta_{2\varepsilon+4+\lambda}$	$2\varepsilon+3$	$2\varepsilon+2$	$2\varepsilon+4$	$2\varepsilon+3$	0	$0 \le \varepsilon \le \psi-2$
$\zeta_{2\varepsilon+5+\lambda}$	$2\varepsilon+6$	$2\varepsilon+3$	$2\varepsilon+5$	$2\varepsilon+2$	0	$0 \le \varepsilon \le \psi-3$
$\zeta_{2\lambda-2\varepsilon-1}$	$2\varepsilon+2$	$2\varepsilon+5$	$2\varepsilon+3$	$2\varepsilon+6$	0	$0 \le \varepsilon \le \psi-3$
$\zeta_{2\lambda-2\varepsilon}$	$2\varepsilon+3$	$2\varepsilon+4$	$2\varepsilon+2$	$2\varepsilon+3$	0	$0 \le \varepsilon \le \psi-2$

We can observe that no two vertices in the inner cycles with same representations. Also there is no single vertex in the inner cycle having same representation with a vertex in the outer cycle, and no two vertices on the outer cycles having same representations. This implies that $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_1\}$, $\theta_2 = \{\eta_3\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = \{\zeta_{\lambda+3}\}$, $\theta_5 = V(P_{\lambda,\lambda-1}) \backslash \{\zeta_1, \zeta_{\lambda+3}, \eta_3, \eta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$ when $\lambda \equiv 0 \ (mod\ 4)$ implying that

$$pd(P_{\lambda,\lambda-1}) \le 5$$

Theorem 2.2 *Let $P_{\lambda,\lambda-1}$ be generalized Petersen graphs with $\lambda \geq 6$ and $\lambda \equiv 2(mod\ 4)$. Then* $pd(P_{\lambda,\lambda-1}) \leq 5$.

Proof. For $\lambda \geq 6$, $\lambda \equiv 2(mod\ 4)$ and for the resolving partition set $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_1\}$, $\theta_2 = \{\eta_3\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = \{\zeta_{\lambda+3}\}$, $\theta_5 = V(P_{\lambda,\lambda-1}) \backslash \{\zeta_1, \zeta_{\lambda+3}, \eta_3, \eta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$.

For $\lambda = 6$, the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are in the Tables 4.

Table 4: Representations of inner and outer vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5
η_1	1	2	4	3	0	η_2	2	1	3	2	0
η_4	4	1	3	2	0	η_5	3	2	2	3	0
η_6	2	3	1	4	0	η_8	2	3	1	2	0
η_9	3	4	2	1	0	η_{10}	4	3	3	2	0
η_{11}	3	4	4	3	0	η_{12}	2	3	3	4	0
ζ_2	3	2	2	1	0	ζ_3	2	1	3	4	0
ζ_4	3	2	4	1	0	ζ_5	4	3	3	4	0
ζ_6	1	4	2	3	0	ζ_7	4	3	1	2	0
ζ_8	1	2	2	3	0	ζ_{10}	3	2	4	3	0
ζ_{11}	2	3	3	2	0	ζ_{12}	3	4	2	3	0

It is clear and easily observable that all the vertices which are $V(P_{\lambda,\lambda-1})$ for $\lambda = 6$ have unique representations with respect to θ. Now for $\lambda > 6$ the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are: $r(\eta_{2\psi+3}|\theta) = (2\psi+1,\ 2\psi,\ 2\psi,\ 2\psi+1,\ 0)$, $r(\eta_{2\psi+2+\lambda}|\theta) = (2\psi+2,\ 2\psi+1,\ 2\psi+1,\ 2\psi,\ 0)$, $r(\zeta_{2\psi+2}|\theta) = (2\psi+1,\ 2\psi,\ 2\psi+2,\ 2\psi-1,\ 0)$, $r(\zeta_{2\psi+3+\lambda}) = (2\psi,\ 2\psi+1,\ 2\psi+1,\ 2\psi,\ 0)$, and remaining in Tables 5, where $\psi = \frac{\lambda-2}{4}$.

Table 5: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	
$\eta_{\varepsilon+1+2\psi}$	$\varepsilon+1+2\psi$	$\varepsilon-2+2\psi$	$2\psi-\varepsilon+2$	$\varepsilon+2\psi-2$	0	$0 \leq \varepsilon \leq 1$
$\eta_{2\psi+\varepsilon+4}$	$2\psi-\varepsilon$	$2\psi+\varepsilon+1$	$2\psi-\varepsilon-1$	$2\psi-\varepsilon+2$	0	
$\eta_{\lambda+\varepsilon+3+2\psi}$	$2\psi-\varepsilon+1$	$2\psi-\varepsilon+2$	$2\psi-\varepsilon+2$	$2\psi+\varepsilon+1$	0	
$\zeta_{2\psi+\varepsilon+3}$	$2\psi-3\varepsilon+2$	$2\psi+1+\varepsilon$	$2\psi+1-\varepsilon$	$2\psi+2-\varepsilon$	0	$0 \leq \varepsilon \leq 1$
$\zeta_{2\psi+1+\lambda+\varepsilon}$	$2\psi+2-\varepsilon$	$2\psi+\varepsilon-1$	$2\psi+1+\varepsilon$	$2\psi+3\varepsilon-2$	0	
$\zeta_{\lambda+2\psi+4+\varepsilon}$	$2\psi-3\varepsilon+1$	$2\psi+2-\varepsilon$	$2\psi-\varepsilon$	$2\psi+1+\varepsilon$	0	

We can observe that no two vertices in the inner cycles with same representations. Also there is no single vertex in the inner cycle having same representation with a vertex in the outer cycle, and no two vertices on the outer cycles having same representations. This implies that $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_1\}$, $\theta_2 = \{\eta_3\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = \{\zeta_{\lambda+3}\}$, $\theta_5 = V(P_{\lambda,\lambda-1}) \backslash \{\zeta_1, \eta_3, \eta_{\lambda+1}, \zeta_{\lambda+3}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$ when $\lambda \equiv 2\ (mod\ 4)$ implying that

$$pd(P_{\lambda,\lambda-1}) \leq 5$$

Theorem 2.3 *Let* $P_{\lambda,\lambda-1}$ *be generalized Petersen graphs with* $\lambda \geq 5$ *and* $\lambda \equiv 1(mod\ 4)$. *Then* $pd(P_{\lambda,\lambda-1}) \leq 4$.

Proof. For $\lambda \geq 5$, $\lambda \equiv 1(mod\ 4)$ and for the resolving partition set $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ where $\theta_1 = \{\eta_2\}$, $\theta_2 = \{\zeta_{\frac{\lambda+3}{2}}\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = V(P_{\lambda,\lambda-1})\backslash\{\zeta_{\frac{\lambda+3}{2}}, \eta_2, \eta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$.

For $\lambda = 5$, the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are in the Tables 6.

Table 6: Representations of inner and outer vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4
η_1	1	3	4	0	η_3	1	2	3	0
η_4	2	1	2	0	η_5	3	2	1	0
η_7	4	3	1	0	η_8	3	2	2	0
η_9	3	3	3	0	η_{10}	2	2	3	0
ζ_1	2	4	3	0	ζ_2	1	2	2	0
ζ_3	2	3	3	0	ζ_5	3	3	2	0
ζ_6	2	2	1	0	ζ_7	3	4	2	0
ζ_8	2	1	3	0	ζ_9	3	4	3	0
ζ_{10}	3	1	2	0					

It is easily observable that all the vertices which are $V(P_{\lambda,\lambda-1})$ for $\lambda = 5$ have unique representations with respect to θ. Now for $\lambda > 5$ the representations of the vertices of $V(P_{\lambda,\lambda-1})$ are: $r(\eta_{j(2\lambda-1)+1}: 0 \leq \varepsilon \leq 1|\theta) = (\varepsilon + 1,\ 2\psi - \varepsilon + 1,\ 4 - \varepsilon,\ 0)$, $r(\eta_{2\psi+1+\varepsilon}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi + \varepsilon - 1,\ 2 - \varepsilon,\ 2\psi + 1 - \varepsilon,\ 0)$, $r(\eta_{2\psi+3}|\theta) = (2\psi + 1,\ 2,\ 2\psi - 1,\ 0)$, $r(\eta_{2+\lambda}) = (4,\ 2\psi + 1,\ 1,\ 0)$, $r(\eta_{\lambda+1+\varepsilon+2\psi}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi + 1,\ 2 + \varepsilon,\ 2\psi + \varepsilon,\ 0)$, $r(\zeta_1) = (2,\ 2\psi + 2,\ 3,\ 0)$, $r(\zeta_{2\psi+1-\varepsilon}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi - \varepsilon,\ 3 - \varepsilon,\ 2\psi + 1 - \varepsilon,\ 0)$, $r(\zeta_{2\psi+3+\varepsilon}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi - \varepsilon + 1,\ 3 - \varepsilon,\ 2\psi - \varepsilon,\ 0)$, $r(\zeta_{2\psi+1+\varepsilon+\lambda}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi + \varepsilon,\ 1 + 3\varepsilon,\ 2\psi + 1,\ 0)$, $r(\zeta_{2\psi+3+\lambda}|\theta) = (2\psi + 1,\ 1,\ 2\psi,\ 0)$, $r(\zeta_{\lambda+2}|\theta) = (3,\ 2\psi + 2,\ 2,\ 0)$, and remaining in Tables 7, where $\psi = \frac{\lambda-1}{4}$.

Table 7: Representations of inner and outer cycle vertices for $0 \leq \phi \leq 2\psi - 3$ and $0 \leq \varepsilon \leq \psi - 1$

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4
$\eta_{\phi+3}$	$\phi + 1$	$2\psi - \phi$	$\phi + 4$	0
$\eta_{\phi+3+\lambda}$	$\phi + 3$	$2\psi - \phi$	$\phi + 2$	0
$\eta_{\lambda-\phi}$	$\phi + 4$	$2\psi - \phi$	$\phi + 1$	0
$\eta_{2\lambda-\phi-1}$	$\phi + 3$	$2\psi - \phi - 1$	$\phi + 4$	0
$\zeta_{2\varepsilon+2}$	$2\varepsilon + 1$	$2\psi - 2\varepsilon$	$2\varepsilon + 2$	0
$\zeta_{\lambda-2\varepsilon+1}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon$	$2\varepsilon + 1$	0
$\zeta_{\lambda+2\varepsilon+3}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon - 1$	$2\varepsilon + 3$	0
$\zeta_{2\lambda-2\varepsilon}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon - 1$	$2\varepsilon + 2$	0
$\zeta_{2\varepsilon+3}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon + 1$	$2\varepsilon + 3$	0
$\zeta_{\lambda-2\varepsilon}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon + 1$	$2\varepsilon + 2$	0
$\zeta_{\lambda+2\varepsilon+4}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon$	$2\varepsilon + 4$	0
$\zeta_{2\lambda-2\varepsilon-1}$	$2\varepsilon + 4$	$2\psi - 2\varepsilon$	$2\varepsilon + 3$	0

We can observe that no two vertices in the inner cycles with same representations. Also there is no single vertex in the inner cycle having same representation with a vertex in the outer cycle, and no two vertices on the outer cycles having same representations. This implies that $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ where $\theta_1 = \{\eta_2\}$, $\theta_2 = \{\zeta_{\frac{\lambda+3}{2}}\}$, $\theta_3 = \{\eta_{\lambda+1}\}$, $\theta_4 = V(P_{\lambda,\lambda-1})\backslash\{\eta_2, \zeta_{\frac{\lambda+3}{2}}, \eta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$ when $\lambda \equiv 1\ (mod\ 4)$ implying that

$$pd(P_{\lambda,\lambda-1}) \leq 4$$

Theorem 2.4 *Let $P_{\lambda,\lambda-1}$ be generalized Petersen graphs with $\lambda \geq 7$ and $\lambda \equiv 3(mod\ 4)$. Then* $pd(P_{\lambda,\lambda-1}) \leq 4$.

Proof. For $\lambda \geq 7$, $\lambda \equiv 3(mod\ 4)$ and for the resolving partition set $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ where $\theta_1 = \{\eta_1\}$, $\theta_2 = \{\zeta_{\frac{\lambda+1}{2}}\}$, $\theta_3 = \{\zeta_{\lambda+1}\}$, $\theta_4 = V(P_{\lambda,\lambda-1})\backslash\{\zeta_{\frac{\lambda+1}{2}}, \eta_1, \zeta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$.

For $\lambda = 7$, the representations of the vertices of $V(P_{\lambda,\lambda-1})$, are in the Tables 8.

Table 8: Representations of inner and outer vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4
η_2	1	3	2	0	ζ_1	1	5	4	0
η_3	2	2	3	0	ζ_2	2	2	1	0
η_4	3	1	4	0	ζ_3	3	3	4	0
η_5	4	2	4	0	ζ_5	4	3	5	0
η_6	4	3	3	0	ζ_6	3	2	2	0
η_7	3	4	2	0	ζ_7	2	5	3	0
η_8	4	4	1	0	ζ_9	2	4	3	0
η_9	3	3	2	0	ζ_{10}	3	1	2	0
η_{10}	4	3	2	0	ζ_{11}	4	4	5	0
η_{11}	4	3	4	0	ζ_{12}	4	1	3	0
η_{12}	3	2	4	0	ζ_{13}	3	4	4	0
η_{13}	2	3	3	0	ζ_{14}	2	3	1	0
η_{14}	1	4	2	0					

It is easily observable that all the vertices which are $V(P_{\lambda,\lambda-1})$ for $\lambda = 7$ have unique representations with respect to θ. Now for $\lambda > 7$ the representations of the vertices of $V(P_{\lambda,\lambda-1})$ are: $r(\eta_{2\psi+2+\varepsilon}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi + \varepsilon + 1,\ 1 + \varepsilon,\ 2\psi + 2,\ 0)$, $r(\eta_{\lambda+1}|\theta) = (4,\ 2\psi + 2,\ 1,\ 0)$, $r(\eta_{2+\lambda+2\psi+\varepsilon}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi - \varepsilon + 2,\ 3 - \varepsilon,\ 2\psi + 2,\ 0)$, $r(\zeta_{2\psi\varepsilon+2\varepsilon+1}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi\varepsilon + 2\varepsilon + 1,\ 2\psi - 2\varepsilon\psi + 3,\ 2\psi\varepsilon - \varepsilon + 4,\ 0)$, $r(\zeta_{2\psi+2+\varepsilon+\lambda}: 0 \leq \varepsilon \leq 1|\theta) = (2\psi + 2,\ 4 - 3\varepsilon,\ 2\psi - 2\varepsilon + 3,\ 0)$, and remaining in Tables 9, where $\psi = \frac{\lambda-3}{4}$.

Table 9: Representations of inner and outer cycle vertices for $0 \leq \phi \leq 2\psi - 1$ and $0 \leq \varepsilon \leq \psi - 1$

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4
$\eta_{\phi+2}$	$\phi + 1$	$2\psi - \phi + 1$	$\phi + 2$	0
$\eta_{\phi+2+\lambda}$	$\phi + 3$	$2\psi - \phi + 1$	$\phi + 2$	0
$\eta_{\lambda-\phi}$	$\phi + 3$	$2\psi - \phi + 2$	$\phi + 2$	0
$\eta_{2\lambda-\phi}$	$\phi + 1$	$2\psi - \phi + 2$	$\phi + 2$	0
$\zeta_{2\varepsilon+2}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon$	$2\varepsilon + 1$	0
$\zeta_{\lambda-2\varepsilon}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon + 3$	$2\varepsilon + 3$	0
$\zeta_{\lambda+2\varepsilon+2}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon + 2$	$2\varepsilon + 3$	0
$\zeta_{2\lambda-2\varepsilon-1}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon + 2$	$2\varepsilon + 4$	0
$\zeta_{2\varepsilon+3}$	$2\varepsilon + 2$	$2\psi - 2\varepsilon + 1$	$2\varepsilon + 3$	0
$\zeta_{\lambda-2\varepsilon}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon + 1$	$2\varepsilon + 2$	0
$\zeta_{\lambda+2\varepsilon+4}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon$	$2\varepsilon + 4$	0
$\zeta_{2\lambda-2\varepsilon-1}$	$2\varepsilon + 4$	$2\psi - 2\varepsilon$	$2\varepsilon + 3$	0
$\zeta_{2\varepsilon+3}$	$2\varepsilon + 3$	$2\psi - 2\varepsilon + 1$	$2\varepsilon + 4$	0

$\zeta_{\lambda-2\varepsilon-1}$	$2\varepsilon+3$	$2\psi-2\varepsilon$	$2\varepsilon+2$	0
$\zeta_{\lambda+2\varepsilon+3}$	$2\varepsilon+3$	$2\psi-2\varepsilon-1$	$2\varepsilon+2$	0
$\zeta_{2\lambda-2\varepsilon}$	$2\varepsilon+2$	$2\psi-2\varepsilon+1$	$2\varepsilon+1$	0

We can observe that no two vertices in the inner cycles with same representations. Also there is no single vertex in the inner cycle having same representation with a vertex in the outer cycle, and no two vertices on the outer cycles having same representations. This implies that $\theta=\{\theta_1,\theta_2,\theta_3,\theta_4\}$ where $\theta_1=\{\eta_1\}$, $\theta_2=\{\zeta_{\frac{\lambda+1}{2}}\}$, $\theta_3=\{\zeta_{\lambda+1}\}$, $\theta_4=V(P_{\lambda,\lambda-1})\backslash\{\eta_1,\zeta_{\frac{\lambda+1}{2}},\zeta_{\lambda+1}\}$ is a resolving partitioning for $V(P_{\lambda,\lambda-1})$ when $\lambda\equiv 3\ (mod\ 4)$ implying that

$$pd(P_{\lambda,\lambda-1})\leq 4$$

1.2 Generalized Peterson Multi-graphs: $P_{2\lambda,\lambda}$

The generalized Petersen graphs are actually multi-graphs denoted as $P_{2\lambda,\lambda}$, having vertex set $V(P_{2\lambda,\lambda})=\{\zeta_\xi,\ \eta_\xi\colon\ 0\leq\xi\leq 2\lambda-1\}$ along the edge set defined as: $E(P_{2\lambda,\lambda})=\{\zeta_\xi\zeta_{\xi+1},\ \zeta_\xi\eta_\xi,\ \eta_\xi\eta_{\xi+\lambda}\colon\ 0\leq\xi\leq 2\lambda-1\}$. The indices for the vertices are taken modulo of 2λ. We use the outer vertices name for the vertices $\zeta_1,\dots,\zeta_{2\lambda-1}$ and inner vertices $\eta_1,\dots,\eta_{2\lambda-1}$.

Theorem 3.1 *Let $P_{2\lambda,\lambda}$ be generalized Petersen multigraphs with $\lambda\geq 4$ and $\lambda\equiv 0(mod\ 4)$. Then* $pd(P_{2\lambda,\lambda})\leq 4$.

Proof. For $\lambda=4$, it is easy to see that $\theta=\{\theta_1,\theta_2,\theta_3,\theta_4\}$ where $\theta_1=\{\zeta_0\}$, $\theta_2=\{\zeta_3\}$, $\theta_3=\{\eta_6\}$, $\theta_4=V(P_{2\lambda,\lambda})\backslash\{\zeta_0,\zeta_3,\eta_6\}$ is a resolving partitioning for $V(P(8,4))$, by given representations in the Tables 10

Table 10: Representations of inner and outer vertices

$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4	$r(.\|\theta)$	θ_1	θ_2	θ_3	θ_4
η_0	1	3	4	0	η_1	2	3	4	0
η_3	3	1	4	0	η_4	2	2	4	0
η_5	3	3	3	0	η_7	2	2	3	0
ζ_1	1	2	3	0	ζ_2	3	1	3	0
ζ_4	3	1	3	0	ζ_5	3	2	2	0
ζ_6	2	3	1	0	ζ_7	1	3	2	0

For $\lambda\geq 4$, $\lambda\equiv 0(mod\ 4)$, we can write $\lambda=4\psi$ where $\psi\geq 1$ and the resolving partition set $\theta=\{\theta_1,\theta_2,\theta_3,\theta_4\}$ where $\theta_1=\{\zeta_\xi\}$, $\theta_2=\{\zeta_{\xi+\lambda-1}\}$, $\theta_3=\{\eta_{\xi+\lambda+2\psi}\}$, $\theta_4=V(P_{2\lambda,\lambda})\backslash\{\zeta_\xi,\zeta_{\xi+\lambda-1},\eta_{\xi+\lambda+2\psi}\}$ is a resolving partitioning for $V(P_{2\lambda,\lambda})$, where $\psi=\frac{\lambda}{4}$.
The representations of the vertices of $V(P_{2\lambda,\lambda})$ are: $r(\eta_{2\psi+1+\xi}|\theta)=(2\psi+1,\ 2\psi-2,\ 3,\ 0)$, $r(\eta_\xi|\theta)=(1,\ 3,\ 2\psi+2,\ 0)$, $r(\eta_{2\psi+\xi-1}|\theta)=(2\psi,\ 2\psi+1,\ 4,\ 0)$, $r(\zeta_{2\psi+\xi}|\theta)=(2\psi+1,\ 2\psi,\ 1,\ 0)$, and in the Table 11.

Table 11: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	
$\eta_{\xi+\varepsilon+1}$	$\varepsilon+2$	$\varepsilon+4$	$2\psi-\varepsilon+2$	0	$0\leq\varepsilon\leq 2\psi-3$
$\eta_{\xi+1+\lambda-\varepsilon}$	$\varepsilon+3$	$\varepsilon+1$	$2\psi-\varepsilon+2$	0	$0\leq\varepsilon\leq 2\psi-2$
$\eta_{\lambda+\xi+\varepsilon}$	$\varepsilon+2$	$\varepsilon+2$	$2\psi-\varepsilon+2$	0	$0\leq\varepsilon\leq 2\psi-1$

$\eta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+2$	$\varepsilon+2$	$2\psi-\varepsilon+1$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{\xi+\varepsilon+1}$	$\varepsilon+1$	$\varepsilon+5$	$2\psi+1-\varepsilon$	0	$0\le\varepsilon\le 2\psi-4$
$\zeta_{\xi+2\psi-2+\varepsilon}$	$2\psi+\varepsilon-2$	$2\psi-\varepsilon+1$	$4-\varepsilon$	0	$0\le\varepsilon\le 2$
$\zeta_{\xi-\varepsilon+\lambda-2}$	$\varepsilon+5$	$\varepsilon+1$	$2\psi-\varepsilon$	0	$0\le\varepsilon\le 2\psi-4$
$\zeta_{\xi+\lambda+\varepsilon}$	$\varepsilon+3$	$\varepsilon+1$	$2\psi+1-\varepsilon$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{2\psi-1+\xi+\lambda+\varepsilon}$	$2\psi+1-\varepsilon$	$2\psi+\varepsilon$	$2-\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+1$	$\varepsilon+3$	$2\psi-\varepsilon$	0	$0\le\varepsilon\le 2\psi-2$

It is easily observable that no two vertices of $P_{2\lambda,\lambda}$ which are in the column 1 of Table 11 have the same representations with respect to the set θ, this implied that θ is a resolving partitioning set for $P_{2\lambda,\lambda}$. Hence

$$pd(P_{2\lambda,\lambda})\le 4.$$

Theorem 3.2 *Let $P_{2\lambda,\lambda}$ be generalized Petersen multigraphs with $\lambda\ge 6$ and $\lambda\equiv 2(mod\ 4)$. Then $pd(P_{2\lambda,\lambda})\le 4$.*

Proof. For $\lambda\equiv 2(mod\ 4)$, we can write $\lambda=4\psi+2$ where $\psi\ge 1$ and the resolving partition set $\theta=\{\theta_1,\theta_2,\theta_3,\theta_4\}$ where $\theta_1=\{\zeta_\xi\}$, $\theta_2=\{\zeta_{\xi+\lambda-1}\}$, $\theta_3=\{\eta_{\xi+\lambda+2\psi}\}$, $\theta_4=V(P_{2\lambda,\lambda})\backslash\{\zeta_\xi,\zeta_{\xi+\lambda-1},\eta_{\xi+\lambda+2\psi}\}$ is a resolving partitioning for $V(P_{2\lambda,\lambda})$, where $\psi=\frac{\lambda-2}{4}$.

The representations of the vertices of $V(P_{2\lambda,\lambda})$ are: $r(\eta_{2\psi+\xi}|\theta)=(2\psi+1,\ 2\psi+2,\ 1,\ 0)$, $r(\eta_{\xi+2\psi+1}|\theta)=(2\psi+2,\ 2\psi+1,\ 4,\ 0)$, $r(\eta_{\lambda+\xi-1}|\theta)=(3,\ 1,\ 2\psi+3,\ 0)$, and in the Table 12.

Table 12: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	
$\eta_{\xi+\varepsilon}$	$\varepsilon+1$	$\varepsilon+3$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi-1$
$\eta_{\xi-2+\lambda-\varepsilon}$	$\varepsilon+4$	$\varepsilon+2$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi-2$
$\eta_{\lambda+\xi+\varepsilon}$	$\varepsilon+2$	$\varepsilon+2$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-1$
$\eta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+2$	$\varepsilon+2$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi$
$\zeta_{\xi+\varepsilon+1}$	$\varepsilon+1$	$\varepsilon+5$	$2\psi+1-\varepsilon$	0	$0\le\varepsilon\le 2\psi-3$
$\zeta_{\xi+2\psi+\varepsilon-1}$	$2\psi+\varepsilon-1$	$2\psi-\varepsilon+2$	$3-\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{\xi+2\psi+\varepsilon+1}$	$2\psi+\varepsilon+1$	$2\psi-\varepsilon$	$3+\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{\xi-\varepsilon+\lambda-2}$	$\varepsilon+5$	$\varepsilon+1$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-3$
$\zeta_{\xi+\lambda+\varepsilon}$	$\varepsilon+3$	$\varepsilon+1$	$2\psi+1-\varepsilon$	0	$0\le\varepsilon\le 2\psi-1$
$\zeta_{2\psi+\xi+\lambda+\varepsilon}$	$2\psi+2-\varepsilon$	$2\psi+\varepsilon+1$	$1+\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+1$	$\varepsilon+3$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-1$

There are no two vertices from the vertex set of graph $P_{2\lambda,\lambda}$, which actually discussed in the column 1 of above Table 12 have the same representations corresponding to the settled set θ, this implied that θ is a resolving partitioning set for the graph $P_{2\lambda,\lambda}$. Hence

$$pd(P_{2\lambda,\lambda})\le 4$$

Theorem 3.3 *Let $P_{2\lambda,\lambda}$ be generalized Petersen multigraphs with $\lambda\ge 5$ and $\lambda\equiv 1(mod\ 4)$. Then $pd(P_{2\lambda,\lambda})\le 5$.*

Proof. For $\lambda\equiv 1(mod\ 4)$, we can write $\lambda=4\psi+1$ where $\psi\ge 1$ and the resolving partition set $\theta=\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5\}$ where $\theta_1=\{\zeta_\xi\}$, $\theta_2=\{\zeta_{\xi+\lambda}\}$, $\theta_3=\{\eta_{\xi+\lambda+2\psi-1}\}$, $\theta_4=\{\eta_{\xi+\lambda+2\psi+1}\}$, $\theta_5=V(P_{2\lambda,\lambda})\backslash\{\zeta_\xi,\zeta_{\xi+\lambda},\eta_{\xi+\lambda+2\psi-1},\eta_{\xi+\lambda+2\psi+1}\}$ is a resolving partitioning for $V(P_{2\lambda,\lambda})$, where $\psi=\frac{\lambda-1}{4}$.

The representations of the vertices of $V(P_{2\lambda,\lambda})$ are: $r(\zeta_{\lambda+\xi-1}|\theta) = (4,\ 1,\ 2\psi+1,\ 2\psi+1,\ 0)$, $r(\eta_{\xi}|\theta) = (1,\ 2,\ 2\psi+2,\ 2\psi+2,\ 0)$, $r(\eta_{2\psi+\xi-1}|\theta) = (2\psi,\ 2\psi+1,\ 1,\ 5,\ 0)$, $r(\eta_{2\psi+\xi}|\theta) = (2\psi+1,\ 2\psi+2,\ 4,\ 4,\ 0)$, $r(\eta_{2\psi+\xi+1}|\theta) = (2\psi+2,\ 2\psi+1,\ 5,\ 1,\ 0)$, and in the Table 13.

Table 13: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	
$\eta_{\xi+\varepsilon+1}$	$\varepsilon+2$	$\varepsilon+3$	$2\psi-\varepsilon+1$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi-3$
$\eta_{\xi-2+\lambda-\varepsilon}$	$\varepsilon+4$	$\varepsilon+3$	$2\psi-\varepsilon+3$	$2\psi-\varepsilon+1$	0	$0\le\varepsilon\le 2\psi-3$
$\eta_{\lambda+\xi+\varepsilon-1}$	$3-\varepsilon$	$2-\varepsilon$	$2\psi-\varepsilon+2$	$2\psi+\varepsilon+2$	0	$0\le\varepsilon\le 1$
$\eta_{\lambda+\xi+\varepsilon+1}$	$3+\varepsilon$	$2+\varepsilon$	$2\psi-\varepsilon$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-3$
$\eta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+2$	$\varepsilon+3$	$2\psi-\varepsilon+3$	$2\psi-\varepsilon+1$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{\xi+\varepsilon+1}$	$\varepsilon+1$	$\varepsilon+4$	$2\psi-\varepsilon$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{\xi+2\psi+\varepsilon}$	$2\psi+\varepsilon$	$2\psi-\varepsilon+1$	$3+\varepsilon$	$3-\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{\xi+\lambda-2-\varepsilon}$	$\varepsilon+5$	$2+\varepsilon$	$2-\varepsilon+2\psi$	$2\psi-\varepsilon$	0	$0\le\varepsilon\le 2\psi-3$
$\zeta_{\xi+\varepsilon+\lambda+1}$	$\varepsilon+4$	$\varepsilon+1$	$2\psi-\varepsilon-1$	$2\psi-\varepsilon+1$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{\xi+\lambda+2\psi+\varepsilon}$	$2\psi-\varepsilon+1$	$\varepsilon+2\psi$	$2+\varepsilon$	$2-\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+1$	$\varepsilon+4$	$2\psi-\varepsilon+2$	$2\psi-\varepsilon$	0	$0\le\varepsilon\le 2\psi-2$

There are no two vertices from the vertex set of graph $P_{2\lambda,\lambda}$, which actually discussed in the column 1 of above Table 13 have the same representations corresponding to the settled set θ, this implied that θ is a resolving partitioning set for the graph $P_{2\lambda,\lambda}$. Hence

$$pd(P_{2\lambda,\lambda}) \le 5,$$

Theorem 3.4 *Let $P_{2\lambda,\lambda}$ be generalized Petersen multigraphs with $\lambda \ge 7$ and $\lambda \equiv 3(mod\ 4)$. Then* $pd(P_{2\lambda,\lambda}) \le 5$.

Proof. For $\lambda \equiv 3(mod\ 4)$, we can write $\lambda = 4\psi+3$ where $\psi \ge 1$ and the resolving partition set $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ where $\theta_1 = \{\zeta_\xi\}$, $\theta_2 = \{\zeta_{\xi+\lambda}\}$, $\theta_3 = \{\eta_{\xi+\lambda+2\psi}\}$, $\theta_4 = \{\eta_{\xi+\lambda+2\psi+2}\}$, $\theta_5 = V(P_{2\lambda,\lambda})\backslash \{\zeta_\xi, \zeta_{\xi+\lambda}, \eta_{\xi+\lambda+2\psi}, \eta_{\xi+\lambda+2\psi+2}\}$ is a resolving partitioning for $V(P_{2\lambda,\lambda})$.

The representations of the vertices of $V(P_{2\lambda,\lambda})$ are: $r(\zeta_{\lambda+\xi-1}|\theta) = (4,\ 1,\ 2\psi+2,\ 2\psi+2,\ 0)$, $r(\eta_{\xi}|\theta) = (1,\ 2,\ 2\psi+3,\ 2\psi+3,\ 0)$, $r(\eta_{2\psi+\xi}|\theta) = (2\psi+1,\ 2\psi+2,\ 1,\ 5,\ 0)$, $r(\eta_{2\psi+\xi+1}|\theta) = (2\psi+2,\ 2\psi+3,\ 4,\ 4,\ 0)$, $r(\eta_{2\psi+\xi+2}|\theta) = (2\psi+3,\ 2\psi+2,\ 5,\ 1,\ 0)$, $r(\eta_{2\psi+\xi+1+\lambda}|\theta) = (2\psi+3,\ 2\psi+2,\ 3,\ 3,\ 0)$, and in the Table 14.

Table 14: Representations of inner and outer cycle vertices

$r(.,\|\theta)$	θ_1	θ_2	θ_3	θ_4	θ_5	
$\eta_{\xi+\varepsilon+1}$	$\varepsilon+2$	$\varepsilon+3$	$2\psi-\varepsilon+2$	$2\psi-\varepsilon+4$	0	$0\le\varepsilon\le 2\psi-2$
$\eta_{\xi-2+\lambda-\varepsilon}$	$\varepsilon+4$	$\varepsilon+3$	$2\psi-\varepsilon+4$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-2$
$\eta_{\lambda+\xi+\varepsilon-1}$	$3-\varepsilon$	$2-\varepsilon$	$2\psi-\varepsilon+3$	$2\psi+\varepsilon+3$	0	$0\le\varepsilon\le 1$
$\eta_{\lambda+\xi+\varepsilon+1}$	$3+\varepsilon$	$2+\varepsilon$	$2\psi-\varepsilon+1$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi-2$
$\eta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+2$	$\varepsilon+3$	$2\psi-\varepsilon+4$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-1$
$\zeta_{\xi+\varepsilon+1}$	$\varepsilon+1$	$\varepsilon+4$	$2\psi-\varepsilon+1$	$2\psi-\varepsilon+3$	0	$0\le\varepsilon\le 2\psi-1$
$\zeta_{\xi+2\psi+\varepsilon+1}$	$2\psi+\varepsilon+1$	$2\psi-\varepsilon+2$	$3+\varepsilon$	$3-\varepsilon$	0	$0\le\varepsilon\le 1$
$\zeta_{\xi+\lambda-2-\varepsilon}$	$\varepsilon+5$	$2+\varepsilon$	$3-\varepsilon+2\psi$	$2\psi-\varepsilon+1$	0	$0\le\varepsilon\le 2\psi-2$
$\zeta_{\xi+\varepsilon+\lambda+1}$	$\varepsilon+4$	$\varepsilon+1$	$2\psi-\varepsilon$	$2\psi-\varepsilon+2$	0	$0\le\varepsilon\le 2\psi-1$

$\zeta_{\xi+\lambda+2\psi+\varepsilon+1}$	$2\psi-\varepsilon+2$	$\varepsilon+2\psi+1$	$2+\varepsilon$	$2-\varepsilon$	0	$0\leq\varepsilon\leq1$
$\zeta_{2\lambda+\xi-1-\varepsilon}$	$\varepsilon+1$	$\varepsilon+4$	$2\psi-\varepsilon+3$	$2\psi-\varepsilon+1$	0	$0\leq\varepsilon\leq2\psi-1$

There are no two vertices from the vertex set of graph $P_{2\lambda,\lambda}$, which actually discussed in the column 1 of above Table 14 have the same representations corresponding to the settled set θ, this implied that θ is a resolving partitioning set for the graph $P_{2\lambda,\lambda}$. Hence

$$pd(P_{2\lambda,\lambda})\leq5.$$

1.3 Conclusion

In this paper we provide the sharp bounds of the partition dimension for the generalized Peterson graph $P_{\lambda,\lambda-1}$, and also for the generalized Peterson multi-graph $P_{2\lambda,\lambda}$. The final conclusion of results respective to the bounds of partition diemsnion are: Theorem 2.3 and 2.4 discussed the partition dimension of the generalized Peterson graph when $\lambda\equiv1,3(mod\ \ 4)$ and resulted in $pd(P_{\lambda,\lambda-1})\leq4$, for Theorem 2.1 and 2.2 discussed the partition dimension of the generalized Peterson graph when $\lambda\equiv0,2(mod\ \ 4)$, which is resulted in $pd(P_{\lambda,\lambda-1})\leq5$. For the generalized Peterson multi-graph $P_{2\lambda,\lambda}$, final conclusion of results respective to bounds of partition diemsnion are: Theorem 3.1 and 3.2 discussed the partition dimension of the generalized Peterson graph when $\lambda\equiv0,2(mod\ \ 4)$, resulted in $pd(P_{2\lambda,\lambda})\leq4$, for Theorem 3.3 and 3.4 discussed the partition dimension of the generalized Peterson graph when $\lambda\equiv1,3(mod\ \ 4)$, which is resulted in $pd(P_{2\lambda,\lambda})\leq5$.

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