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On Double Covers in Hypergraphs

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ABSTRACT

In this paper, we introduce the concept of a symmetric hypergraph which is a generalization of a symmetric graph, and investigates some related properties. Also we give relations between symmetric hypergraphs, locally symmetric graphs and double covers.

KEYWORDS: Hypergraph, Symmetric graph, Symmetric hypergraph, Double cover

1 INTRODUCTION

Let H be a connected graph with vertex set $V(H)$, edge set $E(H)$ and adjacency denoted by $\sim$. An *automorphism* of H is any permutation of the vertices of H preserving adjacency. Under composition the set of all such permutations of $V(H)$ forms a group known as the (full) automorphism group of H and denoted by $Aut(H)$. Let G be a permutation group on a set V and $v \in V$. Denote by $G_v = \{\alpha \in G: v^\alpha = v\}$, the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on V if $G_v = 1$ for every $v \in V$ and *regular* if G is transitive and semiregular.

For a positive integer s, an *s-arc* in a graph H is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices in H such that v_i is adjacent to v_{i-1} for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. Let G be a subgroup of $Aut(H)$. We say that H is (G,s)- *arc transitive*, or (G,s)- *regular* if G acts transitively or regularly on the set of s-arcs of H, and (G,s)- *transitive* if G acts transitively on the set of s-arcs but not on the set of $(s+1)$-arcs of H. In particular, H is said to be *s-arc transitive*, *s-regular* or *s-transitive* if it is $(Aut(H),s)$-arc transitive, $(Aut(H),s)$-regular or $(Aut(H),s)$-transitive, respectively. Also, 0-arc transitive means *vertex transitive*, and 1-arc transitive means *arc transitive* or *symmetric*. Clearly, for $s \geq 2$, an s-arc transitive graph is also $(s-1)$-arc transitive and hence symmetric. A graph H is said to be *edge transitive* if G is transitive on set of its edges and *half transitive* if H is vertex transitive, edge transitive, but not arc transitive. For example, simple cycles are s-arc transitive for all s, the cube graph and the complete graphs K_n are 2-arc transitive (but not 3-arc transitive), and Petersen graph and the complete bipartite graphs $K_{n,n}$ are 3-arc transitive (but not 4-arc transitive). Note that connected arc transitive graphs are necessarily vertex transitive and therefore regular (in the sense that every vertex has the same degree).

Studying s-arc transitive graphs was initiated by a remarkable result of Tutte [10,11] showing that (G,s)-arc transitive graphs of valency three satisfy $s \leq 5$. Later, Weiss [12] proved that if the valency is at least three, then $s \leq 7$.

Given $G \leq Aut(H)$, we say that H is *locally* (G,s)*-arc transitive*, or just *locally* s*-arc transitive*, if H contains an s-arc and given any two s-arc α and β starting at the same vertex v, there exists an element $g \in G_v$ mapping α to β. If all vertices in H have valency at least two, locally (G,s)-arc transitivity implies locally $(G,s-1)$-arc transitivity. In particular, locally 1-arc transitive, or *locally symmetric*, is equivalent to edge transitivity. If H is locally symmetric but not vertex transitive, then it is a bipartite graph and the two parts of the bipartition are orbits of Aut(H). Locally s-arc transitive graphs have been the subject of much investigation see for example the results in [10-12]; and examples of locally s-arc transitive graphs with large values of s are of particular interest. Stellmacher [9] proved that, for a locally (G,s)-arc transitive graph with all vertices having valency at least three, $s \leqslant 9$.

For $s \geq 0$, an s*-arc* in a hypergraph Γ=(V,E) is an alternate sequence of vertices and edges, $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{s-1}, e_s, v_s)$, where each edge e_i is incident to the vertices v_{i-1}, v_i, $1 \leq i \leq s$ and two consecutive vertices or edges are distinct. The hypergraph Γ is s*-arc transitive* if it has an automorphism group which acts transitively on the set of s-arcs. Γ is said to be *symmetric* if its automorphism group acts transitively on the set of 1-arcs.

Let $G \leq Aut(\Gamma)$, then Γ is said to be *locally* s*-arc transitive*, if Γ contains an s-arc and given any two s-arc α and β starting at the same vertex v, there exists an element $g \in G_v$ mapping α to β. We call a hypergraph Γ *locally symmetric*, if it is locally 1-arc transitive.

In this paper we consider a natural extension of symmetric and locally symmetric graphs to the case of hypergraphs. This paper is organised as follows. In section 2, we review several basic concepts and results on hypergraphs and symmetric hypergraphs. In Section 3, we connect the notion of symmetric hypergraphs and locally symmetric graphs. We discuss relations between symmetric hypergraphs and double covers in section 4.

2 BASIC DEFINITIONS AND PRELIMINARIES

We provide some definitions from the theory of hypergraphs. The interested reader should refer to [2]. We consider finite simple hypergraphs. A hypergraph Γ is a pair Γ=(V,E), where V =$\{v_1, v_2, \ldots, v_n\}$ is a set of discrete elements known as vertices and E=$\{e_1, e_2, \ldots, e_m\}$ is a collection of nonempty subsets of V, known as edges. Thus, a edge typically contains any number of vertices. Two vertices u and v are *adjacent* in Γ=(V,E) if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$ holds $e \cap f \neq 0$, we say that e and f are *adjacent*. A vertex v and an edge e are *incident* if $v \in e$. We denote by $\Gamma(v)$ the *neighborhood* of a vertex v, i.e. $\Gamma(v) = \{u \in V: \{u,v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or d*-regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph Γ=(V,E) is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. A hypergraph is known as *uniform* or k*-uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertex is a 2-uniform hypergraph.

For two hypergraphs Γ_1=(V_1,E_1) and Γ_2=(V_2,E_2) a *homomorphism* from Γ_1 into Γ_2 is a mapping $\varphi: V_1 \to V_2$ such that $\varphi(e) = \{\varphi(v_1), \ldots, \varphi(v_r)\}$ is an edge in Γ_2, if $e = \{v_1, \ldots, v_r\}$ is an edge in Γ_1. Note, a homomorphism from Γ_1 into Γ_2 implies also a mapping $\varphi_E: E_1 \to E_2$. A homomorphism φ that is bijective is called an *isomorphism* if holds $\varphi(e) \in E_2$ if and only if $e \in E_1$. We say, Γ_1 and Γ_2 are *isomorphic*, in symbols $\Gamma_1 \cong \Gamma_2$ if there exists an isomorphism between them. An isomorphism from a hypergraph Γ onto itself is an *automorphism*. The *automorphism group* of Γ is denoted by $Aut(\Gamma)$. A hypergraph is *vertex transitive* if its automorphism group acts transitively on the set of vertices. Such a hypergraph is necessarily regular, that is, each vertex is incident to the same number of edges. In the same way a hypergraph is *edge transitive* if its automorphism group acts transitively on the set of edges.

For $s \geq 0$, an *s-arc* in a hypergraph Γ=(V, E) is an alternate sequence of vertices and edges, $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{s-1}, e_s, v_s)$, where each edge e_i is incident to the vertices v_{i-1}, v_i, $1 \leq i \leq s$ and two consecutive vertices or edges are distinct.

The hypergraph Γ is *s-arc transitive* if its automorphism group acts transitively on the set of s-arcs of Γ. Γ is said to be *symmetric* if its automorphism group acts transitively on the set of 1-arcs of Γ. This means transitivity on the set of ordered pairs of adjacent vertices. In the case of graphs, such a graph is said to be *symmetric*. We note the following elementary proposition.

Proposition 2.1. *Let Γ=(V, E) be a symmetric hypergraph such that $|e| \geq 2$ for every $e \in E$. Then*

(i). *Γ is vertex transitive,*

(ii). *Γ is uniform,*

(iii). *Γ is edge transitive.*

Proof. (i) For any $u, v \in V(\Gamma)$, if there are no isolated vertices, then there are edges $e_u, e_v \in E(\Gamma)$ (not necessarily distinct) such that $u \in e_u$ and $v \in e_v$. Moreover, since $|e_u|, |e_v| \geq 2$, there are vertices $u', v' \in V, u \neq u', v \neq v'$, such that $u, u' \in e_u$ and $v, v' \in e_v$. Since Γ is symmetric, there exists $g \in Aut(\Gamma)$ such that $(u, e_u, u')^g = (v, e_v, v')$ and hence $u^g = v$. Therefore Γ is vertex transitive. (ii) Similarly, for every two edges $e_u, e_v \in E(\Gamma)$ there exists $g \in Aut(\Gamma)$ such that $(u, e_u, u')^g = (v, e_v, v')$, where $u, u' \in e_u$ and $v, v' \in e_v$ and $|e_u| = |e_v|$. Hence Γ is uniform. (iii) The proof is straightforward and hence omitted. ■

The *dual* Γ^* of a hypergraph Γ=(V, E) is the hypergraph whose vertices and edges are interchanged, so that $V(\Gamma^*) = \{e_i^* | e_i \in E\}$ and edge set $E(\Gamma^*) = \{v_i^* | v_i \in V\}$ with $v_i^* = \{e_j^* | v_i \in e_j\}$. Clearly, if Γ is d-regular then Γ^* is d-uniform and vice versa. Moreover, by considering the natural action of an automorphism of Γ on its edge set, we have Aut(Γ) = Aut(Γ^*). It is not so difficult to see that a simple hypergraph Γ is edge transitive if and only if its dual is vertex transitive. Thus we have the following result.

Proposition 2.2. *Suppose that Γ be a 2-regular and k-uniform symmetric hypergraph with $k \geq 3$. Then its dual Γ^* is a vertex transitive graph.*

Proof. Let Γ be a symmetric hypergraph, 2-regular and k-uniform with $k \geq 3$. Then its dual Γ^* is a 2-uniform, k-regular graph with $k \geq 3$. Thus Γ^* is a vertex transitive graph. ■

To generalize simple graphs, we say that a hypergraph Γ=(V, E) is *linear* if it is simple and for every pair $e, e' \in E$, $|e \cap e'| \leq 1$. In other words, a hypergraph Γ is linear if every pair of its edges intersects in at most one vertex. It was shown in [8, Proposition 4] that 2-arc transitive hypergraphs, are linear. However, symmetric hypergraphs are not linear in general. Consider the following (counter)example.

Example 2.3. Let Γ=(V, E) be a hypergraph, where V ={0,1,2,3} and
E ={{0,1,2}, {0,1,3}, {0,2,3}, {1,2,3}}.
It is not difficult to see that Γ is symmetric but not linear.

3 SYMMETRIC HYPERGRAPHS AND LOCALLY SYMMETRIC GRAPHS

The *incidence graph* of a hypergraph Γ=(V, E) is a bipartite graph IG(Γ) with a vertex set $S = V \cup E$, and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in e$. In the following theorem we connect the notions of symmetric hypergraphs with locally symmetric graphs.

Theorem 3.1. *Let Γ=(V, E) be a d-regular and k-uniform hypergraph with $d, k \geq 3$ and $\Gamma' :=$ IG(Γ) its incidence graph. If Γ is symmetric, then Γ' is locally symmetric.*

Proof. Assume that G be an automorphism group of Γ which acts transitively on the set of 1-arcs of Γ. We claim that, with the natural action of G on $V(\Gamma')$, the stabilizer G_v, acts transitively on the set of 1- arcs of Γ' starting at a given vertex v. Each 2-arc $(v = v_0, v_1, v_2)$ of Γ' with $v_0 = u_0 \in V$ corresponds to an 1-arc of the form (u_0, e_1, u_1) in Γ where $u_0 = v_0$, $u_1 = v_2$ and $e_1 = v_1$. Since Γ is symmetric, $G_v = G_{u_0}$ acts transitively on the set of 2- arcs of Γ' starting at v.

Similarly, each 1-arc $(v = v_1, v_2)$ of Γ' with $v = e_1 \in E$ corresponds to an 1-arc of the form (u_0, e_1, u_1) in Γ where now $u_1 = v_2$, $e_1 = v_1$ and u_0 is any vertex in $e_1 \backslash u_1$. Thus $G_{u_0, e_1} \leq G_v$ acts transitively on the 1-arcs of Γ' starting at v. ■

4 SYMMETRIC HYPERGRAPHS AND DOUBLE COVERS

A group G acting on a set Ω is called *primitive* if the only partitions of Ω that G preserves are the trivial ones, that is, the whole set or the partition into sets of size one. Let $\Gamma = (V, E)$ be a connected G-vertex transitive hypergraph. The hypergraph Γ is called *G-locally primitive* if for any vertex $v \in V$, the stabiliser G_v acts primitively on the neighbourhood $\Gamma(v)$ of v. The hypergraph is said to be *locally primitive* if it is G-locally primitive for some automorphism group G. On the other hand, Γ is locally primitive if and only if there is a transitive subgroup G of $Aut(\Gamma)$ such that Γ is G-locally primitive. A fundamental problem in investigating locally primitive hypergraphs is to determine the structure of the vertex stabilizer of $Aut(\Gamma)$. Weiss [13] conjectured that for a finite connected G-vertex transitive, G-locally primitive graph Γ and for a vertex $v \in V(\Gamma)$, the size of G_v is bounded above by some function depending only on the valency of Γ (see also [14]).

Let $\Gamma = (V, E)$ be an undirected hypergraph. The *standard double cover* of Γ is the undirected hypergraph $\tilde{\Gamma}$ with vertex set $V \times \{1,2\}$ such that two vertices $(u, 1)$ and $(v, 2)$ are adjacent if and only if u and v are adjacent in Γ. The hypergraph $\tilde{\Gamma}$ is bipartite with bipartite halves $V \times \{1\}$ and $V \times \{2\}$. Thus we have the following proposition.

Proposition 4.1. *Let Γ=(V, E) be an undirected, d-regular and k-uniform hypergraph with $d, k \geq 3$. Then*

(i). *$\tilde{\Gamma}$ is connected if and only if Γ is connected and not bipartite,* (ii). *If Γ is G-locally primitive, then $\tilde{\Gamma}$ is G-locally primitive,*

(iii). *If Γ is symmetric, then $\tilde{\Gamma}$ is locally symmetric.*

Proof. (i) Let Γ be an undirected hypergraph. Then two vertices $(u,1)$ and $(v,2)$ are adjacent if and only if two vertices $(v,1)$ and $(u,2)$ are adjacent. If Γ is also connected, then for any $u,v\in V$ there exists a path P in Γ between u and v. This path lifts to a path in $\tilde{\Gamma}$ between $(u,1)$ and $(v,1)$ if P has even length, and to one between $(u,1)$ and $(v,2)$ if P has odd length. There is a path between $(v,1)$ and $(v,2)$ if and only if v is in an odd cycle in Γ. Thus for an undirected connected hypergraph Γ, $\tilde{\Gamma}$ is connected if and only if Γ contains an odd cycle, that is, if and only if Γ is not bipartite. (ii) Let $G\leq Aut(\Gamma)$, then G also acts as a group of automorphisms of $\tilde{\Gamma}$ with the action $g:(u,i)\mapsto(u^g,i)$. If G is vertex transitive on Γ, then G has two orbits on the set of vertices of $\tilde{\Gamma}$ and the action of G on each orbit is permutationally isomorphic to the action of G on V. Furthermore, $G_v=G_{(v,i)}$ for each $i=1,2$. Then if Γ is undirected, the action of G_v on $\Gamma(v)$ is the same as the action of $G_{(v,i)}$ on $\tilde{\Gamma}((v,i))$. Thus in this case, if Γ is G-locally primitive, $\tilde{\Gamma}$ is also G-locally primitive. (iii) Let $((v,i),e,(v_1,j))$ be an 1-arc in $\tilde{\Gamma}$, then (v,e',v_1) is an 1-arc in Γ. Thus if Γ is symmetric, then $\tilde{\Gamma}$ is locally symmetric. ■

Proposition 4.2. *Let Γ be a G-edge transitive, d-regular and k-uniform bipartite hypergraph with $d,k\geq 3$. Suppose that G has two orbits Δ_1 and Δ_2 on the set of vertices of Γ such that all vertex stabilisers are conjugate. Then Γ is the standard double cover of an orbital digraph Γ' for G on Δ_1, and the bijection which interchanges $(v,1)$ with $(v,2)$ is an automorphism for Γ if and only if Γ' is self-paired.*

Proof. Let Γ be a G-edge transitive, d-regular and k-uniform bipartite hypergraph with $d,k\geq 3$, and suppose that G has two orbits Δ_1 and Δ_2 on $V(\Gamma)$ and all vertex stabilisers are conjugate. Then the two actions of G are permutationally isomorphic and so there exists a bijection $\alpha:\Delta_1\to\Delta_2$ such that $\alpha(v^h)=\alpha(v)^h$ for any $v\in\Delta_1$ and $h\in G$. Let the vertex set of Γ be $\Delta_1\times\{1,2\}$. Define the digraph Γ' with vertex set Δ_1 and arc set such that (v,u) is an arc if and only if two vertices $(v,1)$ and $(u,2)$ are adjacent in Γ. Note that two vertices $(v,1)$ and $(v,2)$ are not adjacent in Γ as $G_{(v,1)}$ acts transitively on edges emerging from $(v,1)$. Then Γ' is a G-vertex transitive and G-arc transitive digraph and so is an orbital digraph for G on Δ_1. Furthermore, Γ is the standard double cover of the digraph Γ'. The bijection $\beta:V(\Gamma)\to V(\Gamma)$ is an automorphism for Γ which interchanges $(v,1)$ and $(v,2)$ for all $v\in\Delta_1$ if and only if for each arc (v,u) in Γ', (u,v) is also an arc. ■

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