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Laplacian energy of the conjugacy class graphs of metabelian groups of order less than 30

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Abstract

Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. The conjugacy class graph $\Gamma(G)$ is defined as: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. In this paper, we compute the Laplacian energy of the conjugacy class graphs of metabelian groups of order less than thirty.

Keywords: Metabelian group, conjugacy class graph, Laplacian energy, eigenvalue
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1 Introduction

A graph Γ is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of Γ called the edges. The vertex-set of Γ is denoted by $V(\Gamma)$, while the edge-set is denoted by $E(\Gamma)$. Let Γ be a graph with set of vertices $V(\Gamma) = \{v_1, \ldots, v_n\}$ and the set of edges $E(\Gamma) = \{e_1, \ldots, e_m\}$. The adjacency matrix of Γ denoted by $A(\Gamma)$, is an $n \times n$ matrix defined as follows: the rows and the columns of $A(\Gamma)$ are the elements of $V(\Gamma)$. If $i \neq j$, then the (i, j)-entry of $A(\Gamma)$ is 0 for nonadjacent and 1 for adjacent vertices v_i and v_j. The (i, i)-entry of $A(\Gamma)$ is 0 for $i \in \{1, \ldots, n\}$. The degree of vertex v_i is denoted by $d_\Gamma(v_i)$ and the degree matrix denoted by $\Delta(\Gamma)$ is defined as $\Delta(\Gamma) = diag(d_\Gamma(v_1), d_\Gamma(v_2), \ldots, d_\Gamma(v_n))$, which is the diagonal matrix of vertex degrees. Then, the Laplacian matrix of Γ is denoted by $L(\Gamma)$ which satisfies $L(\Gamma) = \Delta(\Gamma) - A(\Gamma)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix of Γ. The Laplacian energy of the graph Γ is defined as the sum of the absolute values of the difference between the Laplacian matrix eigenvalues and the ratio of twice the edges number divided by the vertices number, i.e., $LE(\Gamma) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where n is the vertices number and m is the edges number of the graph Γ.
Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. From orders of representatives of conjugacy classes, the following conjugacy class graph $\Gamma(G)$ was introduced in [4]: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. A metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there is an abelian normal subgroup N such that the quotient group $\frac{G}{N}$ is abelian. Clearly, every abelian group is metabelian. It is known that a subgroup of a metabelian group and a direct product of metabelian groups are metabelian. For further information on metabelian groups, see [3]. Recall that HoK denotes the central product of two groups H and K, $K \rtimes H$ is the semidirect product of K and H with normal subgroup K and $K \rtimes_f H$ is the Frobenius group with kernel K and complement H. All further unexplained notations are standard. In this paper, we compute the Laplacian energy of the conjugacy class graphs of metabelian groups of order less than thirty.

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2 Examples and Preliminaries

In this section, we give some examples and preliminary results that will be used in the proof of our main results.

Theorem 2.1. *Any dihedral group is metabelian.*

Proof. Suppose that $D_{2n} = \{a, b | a^n = b^2 = 1, aba = b\}$ denotes a dihedral group of order $2n$. Since the commutator subgroup is the cyclic group $\langle a^2 \rangle$, the result follows. □

Proposition 2.2. *([2]) The multiplicity of 0 as an eigenvalue of $L(\Gamma)$ is equal to the number of connected components of the graph.*

Proposition 2.3. *([1]) The Laplacian matrix of the complete graph K_n has eigenvalues 0 with multiplicity 1 and n with multiplicity $n-1$.*

Now, we give some examples of metabelian groups and find their Laplacian matrices and eigenvalues.

Example 2.4. The alternating group A_3 is an abelian normal subgroup of S_3. Since $\frac{S_3}{A_3} \cong \mathbb{Z}_2$, so the factor group of $\frac{S_3}{A_3}$ is abelian. Thus S_3 is metabelian. Also, the eigenvalue of the Laplacian matrix $\Gamma(S_3)$ is $\mu = 0$ with multiplicity 2 and we have

$$L(\Gamma(S_3)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2.5. Since the center of the quaternion group Q_8 is an abelian normal subgroup of Q_8 such that $|\frac{Q_8}{Z(Q_8)}| = 4$, we deduce that Q_8 is metabelian. Also, the eigenvalues of the Laplacian matrix of $\Gamma(Q_8)$ are $\mu = 0$ with multiplicity 1 and $\mu = 3$ with multiplicity 2 and we have

$$L(\Gamma(Q_8)) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Example 2.6. Since the center of the dihedral group D_{10} is an abelian normal subgroup of D_{10} such that $|\frac{D_{10}}{Z(D_{10})}| = 5$, we deduce that D_{10} is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(D_{10})$ are $\mu = 0$ with multiplicity 2 and $\mu = 2$ with multiplicity 1 and we have

$$L(\Gamma(D_{10})) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 2.7. Since $\mathbb{Z}_3$ is an abelian normal subgroup of $Dic_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ such that the factor group of $\frac{Dic_3}{\mathbb{Z}_3}$ is abelian, Dic_3 is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(Dic_3)$ are $\mu = 0$ with multiplicity 1, $\mu = 1$ with multiplicity 1, $\mu = 3$ with multiplicity 1 and $\mu = 4$ with multiplicity 1 and we have

$$L(\Gamma(Dic_3)) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Example 2.8. Since $\mathbb{Z}_{10}$ is an abelian normal subgroup of Dic_5 such that $|\frac{Dic_5}{\mathbb{Z}_{10}}| = 2$, we deduce that Dic_5 is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(Dic_5)$ are $\mu = 0$ with multiplicity 1, $\mu = 2$ with multiplicity 1, $\mu = 4$ with multiplicity 2 and $\mu = 6$ with multiplicity 2 and we have

$$L(\Gamma(Dic_5)) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Example 2.9. Since $\mathbb{Z}_8$ is an abelian normal subgroup of $M_4(2)$ such that $|\frac{M_4(2)}{\mathbb{Z}_8}| = 2$, we deduce that $M_4(2)$ is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(M_4(2))$ are $\mu = 0$ with multiplicity 1 and $\mu = 6$ with multiplicity 5 and we have

$$L(\Gamma(M_4(2))) = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

Example 2.10. Since $\mathbb{Z}_2 \times \mathbb{Z}_4$ is an abelian normal subgroup of $\mathbb{Z}_4oD_8$ such that $|\frac{\mathbb{Z}_4oD_8}{\mathbb{Z}_2 \times \mathbb{Z}_4}| = 2$, we deduce that $\mathbb{Z}_4oD_8$ is metabelian. Also the eigenvalues of the Laplacian matrix of $\Gamma(\mathbb{Z}_4oD_8)$ are $\mu = 0$ with multiplicity 1 and $\mu = 6$ with multiplicity 5 and we have

$$L(\Gamma(\mathbb{Z}_4oD_8)) = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

Example 2.11. By Theorem 2.1, any dihedral group is metabelian. Therefore, all dihedral groups of order less than 30 such as D_8, D_{10}, D_{12}, D_{14}, D_{16}, D_{18}, D_{20}, D_{22}, D_{24}, D_{26} and D_{28} are metabelian groups.

Example 2.12. Since the direct product of metabelian groups is metabelian, we deduce that $\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $\mathbb{Z}_3 \times Q_8$, $\mathbb{Z}_3 \times S_3$, $\mathbb{Z}_4 \times S_3$, $(\mathbb{Z}_2)^2 \times S_3$, $\mathbb{Z}_2 \times A_4$ and $\mathbb{Z}_2 \times Dic_3$ are metabelian groups.

3 Main results

Theorem 3.1. *Let G be a metabelian group of order less than 30 and $\Delta = (|g_1^G|, |g_2^G|, \ldots, |g_n^G|)$, such that g_i^G are the conjugacy classes of G for $1 \leq i \leq n$. Then the Laplacian energy of $\Gamma(G)$ is given in Table 1.*

Table 1: Laplacian energy of metabelian groups of order less than 30

G	Δ	Orders of representatives of conjugacy classes of G	$LE(\Gamma(G))$
S_3	$(1, 3, 2)$	$(1, 2, 3)$	0
Q_8	$(1, 2, 2, 1, 2)$	$(1, 4, 4, 2, 4)$	4
D_8	$(1, 2, 2, 1, 2)$	$(1, 2, 4, 2, 2)$	4
D_{10}	$(1, 5, 2, 2)$	$(1, 2, 5, 5)$	8/3
A_4	$(1, 3, 4, 4)$	$(1, 2, 3, 3)$	8/3
D_{12}	$(1, 3, 2, 2, 3, 1)$	$(1, 2, 6, 3, 2, 2)$	6
Dic_3	$(1, 3, 1, 2, 3, 2)$	$(1, 4, 2, 3, 4, 6)$	6
D_{14}	$(1, 7, 2, 2, 2)$	$(1, 2, 7, 7, 7)$	6
D_{16}	$(1, 4, 2, 2, 1, 4, 2)$	$(1, 2, 8, 4, 2, 2, 8)$	8
Q_{16}	$(1, 4, 2, 2, 1, 4, 2)$	$(1, 4, 8, 4, 2, 4, 8)$	8
SD_{16}	$(1, 4, 4, 2, 1, 2, 2)$	$(1, 4, 2, 4, 2, 8, 8)$	8
$M_4(2)$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 8, 2, 4, 2, 8, 8, 4, 4, 8)$	10
$\mathbb{Z}_4oD_8$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 2, 2, 4, 2, 4, 4, 4, 4, 2)$	10
$(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_4$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 4, 2, 2, 2, 4, 4, 2, 2, 4)$	10

Table 2: Laplacian energy of metabelian groups of order less than 30

$\mathbb{Z}_2 \times D_8$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 2, 2, 2, 2, 4, 2, 2, 2, 4)$	10
$\mathbb{Z}_2 \times Q_8$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 4, 4, 2, 2, 4, 4, 4, 2, 4)$	10
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 4, 4, 2, 2, 4, 4, 4, 2, 4)$	10
D_{18}	$(1, 9, 2, 2, 2, 2)$	$(1, 2, 9, 3, 9, 9)$	48/5
$\mathbb{Z}_3 \rtimes S_3$	$(1, 9, 2, 2, 2, 2)$	$(1, 2, 3, 3, 3, 3)$	48/5
$\mathbb{Z}_3 \times S_3$	$(1, 3, 1, 2, 3, 1, 2, 3, 2)$	$(1, 2, 3, 3, 6, 3, 3, 6, 3)$	12
D_{20}	$(1, 5, 1, 2, 5, 2, 2, 2)$	$(1, 2, 2, 5, 2, 10, 5, 10)$	32/3
$\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$	$(1, 5, 5, 4, 5)$	$(1, 4, 2, 5, 4)$	6
Dic_5	$(1, 5, 1, 2, 5, 2, 2, 2)$	$(1, 4, 2, 5, 4, 10, 5, 10)$	32/3
$\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$	$(1, 7, 3, 7, 3)$	$(1, 3, 7, 3, 7)$	4
D_{22}	$(1, 11, 2, 2, 2, 2, 2)$	$(1, 2, 11, 11, 11, 11, 11)$	40/3
D_{24}	$(1, 6, 2, 1, 2, 6, 2, 2, 2)$	$(1, 2, 4, 2, 3, 2, 12, 6, 12)$	102/7
Dic_6	$(1, 6, 2, 1, 2, 6, 2, 2, 2)$	$(1, 4, 4, 2, 3, 4, 12, 6, 12)$	102/7
$\mathbb{Z}_3 \rtimes D_8$	$(1, 6, 2, 1, 2, 6, 2, 2, 2)$	$(1, 2, 2, 2, 3, 4, 6, 6, 6)$	102/7
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$(1, 3, 1, 1, 2, 3, 3, 1, 2, 2, 3, 2)$	$(1, 8, 4, 2, 3, 8, 8, 4, 12, 6, 8, 12)$	18
$\mathbb{Z}_2 \times A_4$	$(1, 1, 4, 3, 4, 3, 4, 4)$	$(1, 2, 3, 2, 6, 2, 3, 6)$	32/3
$\mathbb{Z}_4 \times S_3$	$(1, 3, 1, 1, 2, 3, 3, 1, 2, 2, 3, 2)$	$(1, 2, 4, 2, 3, 4, 2, 4, 12, 6, 4, 12)$	18
$\mathbb{Z}_3 \times D_8$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 2, 2, 3, 2, 4, 6, 6, 3, 6, 12, 6, 6, 6, 12)$	16
$(\mathbb{Z}_2)^2 \times S_3$	$(1, 3, 1, 1, 2, 3, 3, 1, 2, 2, 3, 2)$	$(1, 2, 2, 2, 3, 2, 2, 2, 6, 6, 2, 6)$	18
$\mathbb{Z}_3 \times Q_8$	$(1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2)$	$(1, 4, 4, 3, 2, 4, 12, 12, 3, 6, 12, 12, 12, 6, 12)$	16
$\mathbb{Z}_2 \times Dic_3$	$(1, 3, 1, 1, 2, 3, 3, 1, 2, 2, 3, 2)$	$(1, 4, 2, 2, 3, 4, 4, 2, 6, 6, 4, 6)$	18
D_{26}	$(1, 13, 2, 2, 2, 2, 2, 2,)$	$(1, 2, 13, 13, 13, 13, 13, 13)$	120/7
$(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$	$(1, 3, 3, 1, 3, 3, 3, 1, 3, 3, 3)$	$(1, 3, 3, 3, 3, 3, 3, 3, 3, 3)$	14
$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	$(1, 3, 3, 1, 3, 3, 3, 1, 3, 3, 3)$	$(1, 9, 3, 3, 9, 9, 3, 3, 9, 9, 9)$	14
Dic_7	$(1, 7, 1, 2, 7, 2, 2, 2, 2, 2)$	$(1, 4, 2, 7, 4, 14, 7, 14, 7, 14)$	17
D_{28}	$(1, 7, 1, 2, 7, 2, 2, 2, 2, 2)$	$(1, 2, 2, 7, 2, 14, 7, 14, 7, 14)$	17

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