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On the conjugacy class graphs of some dicyclic groups

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Abstract

Let G be a dicyclic group and $\Gamma(G)$ be the attached graph related to its conjugacy classes, which is defined as: the vertices of $\Gamma(G)$ are represented by the non-central conjugacy classes of G and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. In this paper, we calculate the clique number and the girth of $\Gamma(G)$ for dicyclic groups of orders $4p, 8p, 4p^2, 4pq$ and 2^m.

Keywords: Dicyclic group, Conjugacy class, Clique number, Girth
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1 Introduction and Preliminaries

There are many possible ways for associating a graph with a group, for the purpose of investigating these algebraic structures using properties of the associated graph, see for example [[1], [2], [5], [7]]. Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. From orders of representatives of conjugacy classes, the following conjugacy class graph $\Gamma(G)$ was introduced in [8]: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. This graph has been widely studied. See, for instance [3] and [6]. A subset X of the vertices of Γ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and is denoted by $\omega(\Gamma)$. A graph Γ is connected if there is a path between each pair of the vertices of Γ. The length of the shortest cycle in a graph Γ is called the girth of Γ and is denoted by girth(Γ). Recall that $Dic_n = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ is a dicyclic group of order $4n$ $(n \geq 2)$. In this paper, we calculate the clique number and the girth of conjugacy class graph of dicycle groups of orders $4p, 8p, 4p^2, 4pq$ and 2^m, where p and q are two odd primes and m in a natural number.

Lemma 1.1. *[4] The group $G = Dic_n$ has precisely $(n + 3)$ conjugacy classes:*
$\{1\}, \{a^n\}, \{a^i, a^{-i}\}(1 \leq i \leq n-1), \{a^{2j}b, 0 \leq j \leq n-1\}, \{a^{2j+1}b, 0 \leq j \leq n-1\}$.

[1] speaker

Lemma 1.2. *[4] Let $G = Dic_n$ be a dicyclic group of order $4n$ $(n \geq 2)$. If g_i for $1 \leq i \leq n+3$ are the representatives of the conjugacy classes of G, then we have table 1:*

Table 1: Representatives of the conjugacy classes of a dicyclic group of order $4n$

g_i	1	a^n	a^r $(1 \leq r \leq n-1)$	b	ab
$o(g_i)$	1	2	$\frac{2n}{(2n,r)}$	4	4

Lemma 1.3. *Let $G = Dic_n$ be a dicyclic group of order $4n$ $(n \geq 2)$. If $n = p$, where p is an odd prime, then the number of conjugacy classes of G with representatives of type a^r $(1 \leq r \leq n-1)$ are given in Table 2.*

Table 2: Representatives of type a^r in dicyclic groups of order $4p$

$o(a^r)$ $(1 \leq r \leq n-1)$	The number of conjugacy classes of G with representatives of type a^r
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$

Lemma 1.4. *Let $G = Dic_n$ be a dicyclic group of order $4n$ $(n \geq 2)$. If $n = p^2$, where p is an odd prime, then the number of the conjugacy classes of G with representative of type a^r $(1 \leq r \leq n-1)$ are listed in Table 3.*

Table 3: Representatives of type a^r in dicyclic groups of order $4p^2$

$o(a^r)$ $(1 \leq r \leq n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2p^2$	$\frac{\varphi(2p^2)}{2} = \frac{p(p-1)}{2}$
p^2	$\frac{\varphi(p^2)}{2} = \frac{p(p-1)}{2}$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$

Lemma 1.5. *Let $G = Dic_n$ be a dicyclic group of order $4n$ $(n \geq 2)$. If $n = pq$, where p and q are distinct odd primes, then the number of conjugacy classes of G with representatives of type a^r $(1 \leq r \leq n-1)$ are listed in Table 4.*

Table 4: Representatives of type a^r in dicyclic groups of order $4pq$

$o(a^r)$ $(1 \leq r \leq n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2pq$	$\frac{\varphi(2pq)}{2} = \frac{(p-1)(q-1)}{2}$
pq	$\frac{\varphi(pq)}{2} = \frac{(p-1)(q-1)}{2}$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$
q	$\frac{\varphi(q)}{2} = \frac{q-1}{2}$
$2q$	$\frac{\varphi(2q)}{2} = \frac{q-1}{2}$

Lemma 1.6. *Let $G = Dic_n$ be a dicyclic group of order $4n$ $(n \geq 2)$. If $n = 2p$, where p is an odd prime, then the number of conjugacy classes of G with representatives of type a^r $(1 \leq r \leq n-1)$ are given in Table 5.*

Table 5: Representatives of type a^r in dicyclic groups of order $8p$

$o(a^r)$ $(1 \leq r \leq n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$
$4p$	$\frac{\varphi(4p)}{2} = p-1$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
4	$\frac{\varphi(4)}{2} = 1$

In the following examples, we draw the conjugacy class graphs of some dicyclic groups.

Example 1.7. By Table 2 and Table 5, the conjugacy class graphs of dicyclc groups of orders 24 and 28 are given in Figure 1 and Figure 2, respectively.

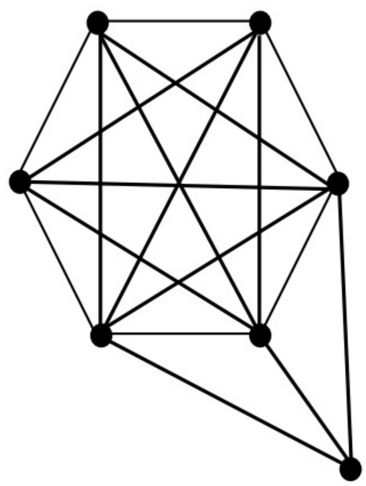

Figure 1: Conjugacy class graph of Dic_6

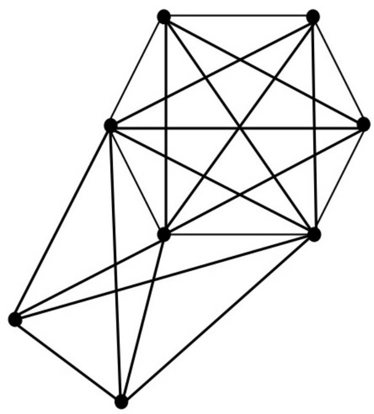

Figure 2: Conjugacy class graph of Dic_7

2 Main results

Theorem 2.1. *Let $G = Dic_n$ be a dicyclic group of order $4n$.*

i) If $n = p$, where p is odd prime, then $\omega(\Gamma(Dic_p)) = p-1$ for $p \geq 5$ and $\omega(\Gamma(Dic_3)) = 3$. Also, $girth(\Gamma(Dic_p)) = 3$ for $p \geq 3$.

ii) *If* $n = p^2$*, where* p *is an odd prime, then* $\omega(\Gamma(Dic_{p^2})) = p^2 - 1$ *and* $girth(\Gamma(Dic_{p^2})) = 3$.

iii) *If* $n = pq$*, where* p *and* q *are distinct odd primes, such that* $p > q$*, then* $\omega(\Gamma(Dic_{pq})) = q(p-1)$ *and* $girth(\Gamma(Dic_{pq})) = 3$.

iv) *If* $n = 2p$*, where* p *is an odd prime, then* $\omega(\Gamma(Dic_{2p})) = 2(p-1)$ *for* $p \geq 7$ *and* $\omega(\Gamma(Dic_{2p})) = \frac{3p+3}{2}$ *for* $p = 3, 5$*. Also* $girth(\Gamma(Dic_{2p})) = 3$ *for* $p \geq 3$.

v) *If* $n = 2^m$*, where* m *is a positive integer, then* $\omega(\Gamma(Dic_{2^m})) = 2^m + 1$ *and* $girth(\Gamma(Dic_{2^m})) = 3$.

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