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On a class of non-chordal graphs with only clique roots

Hossein Teimoori Faal[1]
Allameh Tabataba'i University, Tehran, Iran

Abstract

Polynomial with only real roots play an essential role in mathematics and computer science. The ordinary generating function of the number of complete subgraphs of a graph G is called the clique polynomial of G. Moreover, the real root of the clique polynomial of G is said to be the *clique root* of G . In answering to an open question regarding the class of non-chordal graphs with only clique roots already proposed by Teimoori, we introduce an interesting class of graphs with this unique property. The main ideas behind the proofs are based on graph-theoretical interpretations of higher derivatives of clique polynomials and the intermediate value theorem from calculus. We finally conclude the paper with a couple of possible directions for future research works.

Keywords: Non-chordal graphs, Clique roots, Clique polynomials, higher derivatives, intermediate value theorem.
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1 Introduction

The usage of algebraic tools in finite graphs has a long history in the development of the classical and modern graph theory. Among many interesting research areas of the algebraic graph theory the field of graph polynomials has attracted the attention of many mathematicians and computer scientists in recent years. More importantly, the class of *real-rooted polynomials* has been extensively investigated in recent decades.

On of the main reasons behind this interest is due to the well-known and classic result of Issac Newton [7] which asserts that the class of real-rooted polynomials $P_n(x) = \sum_{i=0}^{n} a_i x^i$ are *log-concave*; that is, their coefficients satisfy the following inequalities

$$a_i^2 \geq a_{i+1}a_{i-1}, \quad (i \geq 1).$$

We only note here that the problem of classifying the class of *log-concave polynomials* is a very challenging problem with many interesting open problems and questions. The recent solution of *Rota's conjecture* [6] on the *log-concavity* of a class of polynomials related to matroid theory is a very strong evidence for the importance of the above claim.

A complete subgraph of a graph G is called a *clique* of G. The ordinary generating function of the number of cliques of G denoted by $C(G, x)$ is said to be the *clique polynomial* of G. We will also call the real root of $C(G, x)$ the *clique root* of G.

Hajiabolhassan and Mehrabadi [2] have shown that any finite, simple and undirected graph has (at least) a clique root. Moreover, they showed that the class of K_3-free graphs has only clique roots. In this line of research, Teimoori [3] extend the previous results of [2] by showing that the class of connected K_4 - free *chordal* graphs has only clique roots. He [4] further extended his result by showing that the class of *bi-connected* K_5 -free chordal graphs has the same real-rootedness property. In the same paper, the following key open question has been proposed:

[1]speaker

Which classes of 2-connected K_5-free non-chordal graphs have only clique roots?

In this paper, motivated by the above question and more imprtantly continuing the ongoing project of identifying the classes of graphs with only clique roots, we start a new step by introducing a class of *non-chordal* graphs with only clique roots. Indeed, we introduce an interesting class of graphs that we will call them l - *triangulated* graphs. For a positive integer l, a graph G is said to be l-*triangulated* if each edge of G belongs to at most $l+1$ triangles of G. In this paper, we will focus on the special case of $l=1$ and will call this class of graphs the *triangulated* graphs. One of the simple example of such graphs is the *wheel graph* W_5 . We note that $C(W_5,x)=1+6x+10x^2+5x^3$. It is not hard to see that $C(W_5,x)=(1+x)[1+5x(x+1)]$, which immediately implies that not only the wheel graph W_5 has clique root $r=-1$ but also it has only clique roots. One can easily see that W_5 is an example of the *non-chordal* triangulated graph.

2 Basic Definitions and Notations

Throughout this paper we assume that all graphs are finite, simple and undirected. For terminologies which are not given here, the interested reader can consult the book [1] .
Let $G=(V,E)$ be a graph with the vertex set V and the edge set E. An open *neighborhood* of a vertex v, denoted by $N_G(v)$ is defined to be a set of vertices *adjacent* to v. The degree of the vertex is the cardinality of the set $N_G(v)$. A *complete* subgraph Q of G is called a *clique* of G. A clique on k vertices is called a k-clique of G. The number of k-cliques of G is denoted by $c_k(G)$. The set of k-cliques of G will be denoted by $\Delta_k(G)$. The ordinary generating function of the number of k-cliques of G is called the *clique polynomial* of G and is denoted by $C(G,x)$. In other words, we have

$$C(G,x)=1+\sum_{k=1}^{\omega(G)} c_k(G)x^k,$$

in which $\omega(G)$ is the number of the vertices of the largest clique of G. The parameter $\omega(G)$ is called the *clique number* of G. A *chordal* graph is a graph in which every cycle of length greater than three has a *chord* (an edge connecting two non-adjacent vertices on the cycle).
From here on, we will use the notation $N(e)=N(u)\cap N(v)$ for a given edge $e=\{u,v\}$. The following vertex and edge-recurrence relations are known for clique polynomials [2] .

$$\begin{aligned} C(G,x) &= C(G-v,x)+xC(G[N_G(v)],x), \qquad (v\in V(G)),\\ C(G,x) &= C(G-e,x)+x^2C(G[N_G(e)],x), \qquad (e\in E(G)). \end{aligned}$$

The following combinatorial interpretation of the clique polynomial and related polynomials is given in [5].

Theorem 2.1. *Let $G=(V,E)$ be a graph. Then, we have*

$$\frac{d}{dx}C(G,x)=\sum_{v\in V(G)} C(G[N_G(v)],x). \tag{1}$$

Furthermore, the author of this paper [4] has extended the above result by giving the following first-ever graph-theoretical interpretation of the *second derivative* of clique polynomials.

Theorem 2.2. *Let $G=(V,E)$ be a graph. Then, we have*

$$\frac{1}{2}\frac{d^2}{dx^2}C(G,x)=\sum_{e\in E(G)} C(G[N_G(e)],x). \tag{2}$$

3 Main results

We first present the main definition of this paper which is the new class of l-triangualted graphs.

Definition 3.1. Let $l > 0$ be a positive integer. The class of *l-triangulated* graphs is the class of all graphs G in which each edge of G lies in *at most* $(l+1)$ triangle of G. In particular, when $l = 1$, we simply call them the class of *triangulated* graphs.

Example 3.2. The wheel W_5 is a simple example of a triangulated graph which is clearly not a chordal graph.

We also need the following well-known result due to *Euler*.

Lemma 3.3. *For a planar graph G with n vertices, m edges and f faces, we have*

$$n - m + f = 2. \tag{3}$$

In particular, for triangulated graph $f = t + 1$, where t denotes the number of triangles of G.

An edge e is called *isolated* if $N(e) = \emptyset$. One of the main results of this paper is the following.

Theorem 3.4. *The class of K_4-free* triangulated *graphs without isolated edges has only clique roots. In particular, this class has always the clique root $r = -1$.*

The Sketvh of the Proof. We first note that having a clique root $r = -1$ is the direct consequence of Lemma 3.3. Now, considering the fact that in a triangulated graph without isolated edges, the neighborhood of each vertex is a path (class 1) or a cycle (class 2), the sum in the right-hand side of equation (1) can run into two disjoint set of vertices (class 1 and class 2) such that the result of evaluating both sums at $x = -\frac{1}{2}$ is not *positive*. Hence, we get $\frac{d}{dx}C(G, -\frac{1}{2}) \leq 0$. Thus, by intermediate value theorem, we immediately get the desired result.

□

By some slight modifications of Theorem 3.4, we can get the following interesting result.

Theorem 3.5. *The class of K_4-free triangulated graphs in which no isolated edge lies in any induced cycle of G has only clique roots. In particular, this class has always a clique root $r = -1$.*

We can even obtain a stronger result, as follows.

Theorem 3.6. *Let G be a K_4-free triangulated graph. Then, the graph G has only clique roots.*

4 Concluding Remarks

In this paper, we extend the result of previous research papers regarding the class of graphs with only clique roots from chordal graphs to non-chordal ones. The main ingradients of the proofs are the graph-theoretical interpretations of higher-order derivatives of graph polynomials and the intermediate value theorem as important tools in the theory of *real-rooted polynomials.*

The first immediate research step is to generalize the result of this paper for the class of K_5-free graphs. The first issue arising in this direction is that not all 2-connected triangulated K_5-free graphs have only clique roots. It can be easily seen that the *non-chordal* graph G_1 obtained by attaching an edge to one of the vertices of K_4 and also attaching a C_4 to another vertex of it has not only clique roots. Indeed, we have $C(G_1, x) = 1 + 8x + 11x^2 + 4x^3 + x^4$ which has at most two real roots beacuse of the fact that the quadratic equation $\frac{1}{2}\frac{d^2}{dx^2}C(G_1, x) = 11 + 12x + 6x^2$ has no real roots. Thus, we come up with the following conjecture.

Conjecture1. Let G be a K_5-free triangualted graph without isolated edges. Then, G has only clique roots.

Another possible project is to extend the results of this paper for the general class of l-triangulated $(l > 1)$ graphs.

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Email: hossein.teimoori@atu.ac.ir