6th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 17-18, 2021

Numerical study on system of fractional Fredholm integro–differential equations via the second Chebyshev wavelets

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Abstract

In this paper, a numerical method for approximating the solutions of system of fractional-order Fredholm integro–differential equations has been proposed. This method is based on the second Chebyshev wavelets and the block pulse functions. The proposed methods reduce the system of fractional-order Fredholm integro–differential equations to a system of algebraic equations that can be easily solved by any usual numerical methods. Finally, a numerical example show the effectiveness and feasibility of this method.

Keywords: Second Chebyshev wavelets, Fredholm integro–differential equations, Block pulse function, operational matrices
Mathematics Subject Classification [2010]: 34A08, 26A33

1 Introduction

In this paper, we solve a system of fractional Fredholm integro–differential equations in the following form:

$$\begin{cases} D^{\alpha_1}u(t) = \lambda_1 \int_0^1 k_1(t,s)v(s)ds + f(t) \\ D^{\alpha_2}v(t) = \lambda_2 \int_0^1 k_2(t,s)u(s)ds + g(t) \end{cases} \qquad u(0) = 0, v(0) = 0 \tag{1}$$

Where $u(t), v(t)$ are unknown functions, functions $f(t), g(t), k_1(t,s)$ and $k_2(t,s)$ are known and λ_1, λ_2 are real constants. Here $\alpha_1, \alpha_2 \in [0,1]$ and $D^{\alpha_1}, D^{\alpha_2}$ denotes the Caputo fractional derivative.

In this section, some notations, definitions and properties are provided about fractional calculus and the second Chebyshev wavelets.

1.1 Fractional calculus

The fractional operators and their properties are defined as following.

Definition 1.1. The Caputo fractional derivative of order α, of the function $y(t)$ is defined as

$$D^{\alpha}y(t) = \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{y^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau,$$

where $k-1 < \alpha \le k, k \in \mathbb{N}$[5].

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Definition 1.2. The Riemann–liouville fractional integral of order α, is given by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{y(\tau)}{(t-\tau)^{\alpha-1}}d\tau,$$

where $\Gamma(.)$ is the Gamma function and $m-1 < \alpha \leq m, m \in \mathbb{N}$[1].

The relationship between the Caputo fractional derivative operator and Riemann–Liouville fractional integral operator is given by the following expressions [7]:

$$\begin{aligned} D^{\alpha}I^{\alpha}y(t) &= y(t), \\ I^{\alpha}D^{\alpha}y(t) &= y(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}y^{(k)}(0). \end{aligned} \tag{2}$$

1.2 The block-pulse functions and operational matrix of the fractional integration

In this section, the block pulse functions (BPFs) and their properties are investigate. An m'-set of BPFs on the interval $[0,1)$ is defined as

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m'} \leq t < \frac{i}{m'}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 1, ..., m'$. The following properties of BPFs will be considered[6]:

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \qquad \int_0^1 b_i(t)b_j(t)dt = \begin{cases} \frac{1}{m'}, & i = j, \\ 0, & i \neq j, \end{cases}$$

Let $B_{m'}(t) = [b_1(t), b_2(t), ..., b_{m'}(t)]^T$, hence the BPFs operational matrix of fractional integration F^{α} is given by

$$I^{\alpha}B_{m'}(t) = F^{\alpha}B_{m'}(t),$$

where

$$F^{\alpha} = \frac{1}{m^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, $k = 1, 2, ..., m$[2].

2 The second Chebyshev wavelets

Definition 2.1. The second Chebyshev wavelets are defined on the interval $[0,1)$ as:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 1, 2, \dots, 2^{k-1}, m = 0, 1, \dots, M-1, k$ and M are positive integers and coefficient $\sqrt{\frac{2}{\pi}}$ is used for orthonormalitly[4]. The function $U_m(t)$ is the second Chebyshev polynomial of degree m. Note that, These polynomials are defined on the interval $[-1,1]$ by the recurrence

$$U_0(t) = 1, \qquad U_1(t) = 2t, \qquad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t),$$

where $m = 1, 2, \dots M$ [6].

A function $f \in L^2([0,1])$ can be approximate in terms of the second Chebyshev wavelets as[3]

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t) = \hat{f}(t),$$

Where

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1(M-1)}(t), \psi_{20}(t), \dots, \psi_{2(M-1)}(t), \dots, \psi_{2^{k-1}0}(t), \dots, \psi_{2^{k-1}(M-1)}(t)]^T,$$
$$C = [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^T.$$

We define the second Chebyshev wavelets matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'} = [\Psi(\frac{1}{2m'}), \Psi(\frac{3}{2m'}), \dots, \Psi(\frac{2m'-1}{2m'})],$$

where $m' = 2^{k-1}M$.

The second Chebyshev wavelets can be expanded in terms of BPFs as

$$\Psi(t) = \Phi_{m'\times m'}B_{m'}(t).$$

Let

$$I^\alpha\Psi(t) \approx P^\alpha{}_{m'\times m'}\Psi(t), \qquad P^\alpha{}_{m'\times m'} = \Phi F^\alpha \Phi^{-1} \tag{3}$$

where I^α is the Riemann-Liouville fractional integral operator of order α. The matrix $P^\alpha_{m'\times m'}$ is called the second Chebyshev wavelets operational matrix of fractional integration[6].

3 Method analysis

For solving this system, now we approximate $D^\alpha u(t)$, $D^\alpha v(t)$, $f(t)$, $g(t)$ and $k_i(t,s)$ for $i = 1,2$ in terms of the second Chebyshev wavelets as following

$$D^\alpha u(t) \simeq C_1^T\Psi(t), \qquad D^\alpha v(t) \simeq C_2^T\Psi(t), \tag{4}$$
$$f(t) \simeq F^T\Psi(t), \qquad g(t) \simeq G^T\Psi(t), \tag{5}$$
$$k_i(t,s) \simeq \Psi^T(t)K_i\Psi(s) \qquad i = 1,2. \tag{6}$$

From Eqs.(2), (3) and (4), we obtain

$$u(t) = I^{\alpha_1}D^{\alpha_1}u(t) \simeq I^{\alpha_1}C_1^T\Psi(t) = C_1^TP^{\alpha_1}\Psi(t), \tag{7}$$
$$v(t) = I^{\alpha_2}D^{\alpha_2}v(t) \simeq I^{\alpha_2}C_2^T\Psi(t) = C_2^TP^{\alpha_2}\Psi(t). \tag{8}$$

From Eqs(6), (7) and (8) and $\int_0^1 \Psi(s)\Psi(s)^Tds = D$, we have

$$\begin{aligned}\int_0^1 k_1(t,s)v(s)ds &= \int_0^1 \Psi^T(t)K_1\Psi(s)\Psi(s)^TP^{\alpha_2 T}C_2ds = \Psi^T(t)K_1\int_0^1 \Psi(s)\Psi(s)^TdsP^{\alpha_2 T}C_2\\ &= \Psi^T(t)K_1DP^{\alpha_2 T}C_2 = C_2^TP^{\alpha_2}D^TK_1^T\Psi(t),\end{aligned} \tag{9}$$
$$\begin{aligned}\int_0^1 k_2(t,s)u(s)ds &= \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi(s)^TP^{\alpha_1 T}C_1ds = \Psi^T(t)K_2\int_0^1 \Psi(s)\Psi(s)^TdsP^{\alpha_1 T}C_1\\ &= \Psi^T(t)K_2DP^{\alpha_1 T}C_1 = C_1^TP^{\alpha_1}D^TK_2^T\Psi(t).\end{aligned} \tag{10}$$

By substituting the Eqs. (4), (5), (9) and (10) into (1), we get

$$\begin{cases} C_1^T\Psi(t) = \lambda_1C_2^TP^{\alpha_2}D^TK_1^T\Psi(t) + F^T\Psi(t) \\ C_2^T\Psi(t) = \lambda_2C_1^TP^{\alpha_1}D^TK_2^T\Psi(t) + G^T\Psi(t)\end{cases} \tag{11}$$

Dispersing Eq.(11), we obtain

$$\begin{cases} C_1^T = \lambda_1 C_2^T P^{\alpha_2} D^T K_1^T + F^T \\ C_2^T = \lambda_2 C_1^T P^{\alpha_1} D^T K_2^T + G^T \end{cases} \tag{12}$$

By solving system (12),we can get C_1 and C_2 . Then substituting them into (7) and (8) , the unknown solutions can be obtained.

4 Numerical example

To demonstrate the efficiency the of this method, we consider the following a numerical example.

Example 4.1. Consider the system of fractional Fredholm integro-differential equations

$$\begin{cases} D^{0.3}u(t) = \int_0^1 (s+t)v(s)ds + f(t) \\ D^{0.4}v(t) = \int_0^1 (t-s)u(s)ds + g(t) \end{cases} \qquad u(0) = 0, v(0) = 0$$

where

$$f(t) = \frac{200}{119\Gamma(0.7)} t^{\frac{17}{10}} - (\frac{t}{2} - \frac{1}{3}), \quad g(t) = \frac{5}{3\Gamma(0.6)} t^{\frac{3}{5}} - (\frac{t}{3} + \frac{1}{4}).$$

The exact solutions of the problem are $u(t) = t$ and $v(t) = t^2$. The absolute errors for $u(t)$ and $v(t)$ are listed Table 1 and 2 shows the absolute errors for different values of t.

Table 1: Absolute error for $M = 3$ and $k = 2, 4, 6$ of $u(t)$ in Example 4.1

t	$M=3, k=2$	$M=3, k=4$	$M=3, k=6$
0	$2.3921e-03$	$1.4923e-04$	$9.5090e-06$
0.1	$3.5481e-03$	$3.0580e-04$	$2.4198e-05$
0.2	$4.4939e-03$	$3.6210e-04$	$2.9457e-05$
0.3	$5.2296e-03$	$4.1360e-04$	$3.4447e-05$
0.4	$5.7551e-03$	$4.6423e-04$	$3.9480e-05$
0.5	$6.2756e-03$	$5.1532e-04$	$4.4630e-05$
0.6	$6.8124e-03$	$5.6732e-04$	$4.9918e-05$
0.7	$7.3564e-03$	$6.2043e-04$	$5.5350e-05$
0.8	$7.9079e-03$	$6.7471e-04$	$6.0923e-05$
0.9	$8.4666e-03$	$7.3017e-04$	$6.6633e-05$

Table 2: Absolute error for $M = 3$ and $k = 2, 4, 6$ of $v(t)$ in Example 4.1

t	$M=3, k=2$	$M=3, k=4$	$M=3, k=6$
0	$2.3912e-02$	$6.1403e-03$	$1.5430e-03$
0.1	$1.2955e-02$	$6.4551e-04$	$7.3894e-05$
0.2	$5.5907e-03$	$3.9567e-04$	$4.2418e-05$
0.3	$1.8204e-03$	$2.7351e-04$	$3.0380e-05$
0.4	$1.6438e-03$	$2.1668e-04$	$2.4534e-05$
0.5	$1.7180e-03$	$1.9167e-04$	$2.1706e-05$
0.6	$1.7370e-03$	$1.8591e-04$	$2.0685e-05$
0.7	$1.8537e-03$	$1.9287e-04$	$2.0885e-05$
0.8	$2.0682e-03$	$2.0929e-04$	$2.1983e-05$
0.9	$2.3804e-03$	$2.3309e-04$	$2.3780e-05$

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