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Generalized Implicit Equilibrium Problems with Application in Game Theory

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ABSTRACT

In this paper, first a brief history of equilibrium problems(EP) and generalized implicit vector equilibrium problems(GIVEP) are given. Then some existence theorems for GIVEP are presented. As applications of our results, we derive some suitable conditions for existing a normalized Nash equilibrium problems when the number of players are finite and the abstract case, that is infinite players.

KEYWORDS: Equilibrium Problems, Game Theory, Generalized Nash equilibrium problem, KKM –mapping,

1 INTRODUCTION

Game theory has been applied during the last two decades to an ever increasing number of important practical problems in economics, industrial organization, business strategy, finance, accounting, market design and marketing; including antitrust analyses, monetary policy, and firm restructuring.
Game Theory is a method of modeling the interaction between two or more players in a situation with particular rules and expected outcomes.
It is helpful in many fields, but mainly as a tool in economics. Game Theory helps with the fundamental analysis of industries and the interactions between two or more companies.
Game Theory revolutionized economics and business analysis by addressing critical issues in the popular mathematical models. For example, neoclassical economists struggle to account for the concept of imperfect competition fully. Game Theory improves on that by switching the focus from constant equilibrium to analyzing the actual market process.
An essential concept within Game Theory is the Nash Equilibrium, which represents a stable state in a game, also known as a ‘no regrets’ state.

Definition 1.1(Game Theory Definitions) Any time we have a situation with two or more players that involve known payouts or quantifiable consequences, we can use game theory to help determine the most likely outcomes. Let's start by defining a few terms commonly used in the study of game theory:

- **Game**: Any set of circumstances that has a result dependent on the actions of two or more decision-makers (players)
- **Players**: A strategic decision-maker within the context of the game
- **Strategy**: A complete plan of action a player will take given the set of circumstances that might arise within the game
- **Payoff**: The payout a player receives from arriving at a particular outcome (The payout can be in any quantifiable form, from dollars to utility.)
- **Information set**: The information available at a given point in the game (The term information set is most usually applied when the game has a sequential component.)
- **Equilibrium**: The point in a game where both players have made their decisions and an outcome is reached.

The implicit vector equilibrium problem (IVEP) was introduced by Huang et al. [8] as follows:

Given a vector valued bifunction $f: K \times K \to Y$ and $g: K \to K$, find $x \in K$ such that

$$f(g(x), y) \not\subset - intC, \quad \forall y \in K, \tag{1}$$

where X, Y are two topological vector spaces and K is a nonempty subset of X.denotes the space of all continuous linear operators from X to Y,

If $T: K \to L(X, Y), \theta: K \times K \to X$, and $g: K \to K$, then (IVEP) reduces to the generalized vector variational inequality (GWI) of finding $x \in K$ such that

$$\langle T(g(x)), \theta(y, g(x)) \rangle \not\subset - intC(x), \quad \forall y \in K, \tag{2}$$

where $L(X, Y)$ denotes the space of all continuous linear operators from X to Y, $\langle T(z), y \rangle$ denotes the evaluation of the linear operator $T(z)$ at y.

The generalized vector equilibrium problem was first introduced in 1997 [1] as follows.

Let K a nonempty, closed, and convex subset of topological vector space (tvs) X, C a closed and convex cone in Y with $intC \neq \emptyset$. Let $F: K \times K \to 2^Y$ be a set-valued mapping. The generalized vector equilibrium problem (GVEP) for F consists in finding $x \in K$ such that

$$F(x, y) \nsubseteq -intC, \quad \forall y \in K. \tag{3}$$

The authors of [16] considered the generalized implicit operator equilibrium problem (GIOEP) which consists of finding $f^* \in K$ such that

$$F(h(f^*), g) \nsubseteq -intC(f^*), \forall g \in K, \tag{4}$$

where $F: K \times K \longrightarrow 2^Y$ is a set-valued mapping, $h: K \longrightarrow K$ is a mapping, X and Y are two Hausdorff topological vector spaces, $L(X, Y)$ is the space of all continuous linear operators from X to Y, $K \subseteq L(X, Y)$ is a nonempty convex set, $C: K \longrightarrow 2^Y$ is a set-valued mapping such that for each $f \in K$, $C(f)$ is a closed and convex cone in Y with $intC(f) \neq \emptyset(intC(f)$ is the interior of $C(f))$, 2^Y denotes the set of all non-empty subsets of Y. This paper is motivated and inspired by the recent paper [8] and its aim is to extend the results given in [8] to the setting of Hausdorff topological vector spaces with mild assumptions and relaxing some conditions. In the rest of this section we recall some definitions and results that we need in the next section.

A subset C of Y is called a pointed and convex cone if and only if $C + C \subseteq C, tC \subseteq C, \forall t \geq 0$, and $C \cap -C = \{0_Y\}$ (see, for instance, [1, 5]) The domain of a set-valued mapping $W: X \longrightarrow 2^Y$ is defined as

$$D(W) = \{x \in X: W(x) \neq \emptyset\} \tag{5}$$

and its graph is defined as

$$Gr(W) = \{(x, z) \in X \times Y: z \in W(x)\}. \tag{6}$$

Also W is said to be closed if its graph, that is, $Gr(W)$, is a closed subset of $X \times Y$.

A set-valued mapping $T: X \longrightarrow 2^Y$ is called upper semicontinuous (in short $u.s.c.$) at $x_0 \in X$ if for every open set $V \subseteq Y$ containing $T(x_0)$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \subseteq V$, for all $u \in U$. The mapping T is said to be lower semicontinuous (in short $l.s.c.$) at $x_0 \in X$ if for every open set $V \subseteq Y$ with $T(x_0) \cap V \neq \emptyset$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \cap V \neq \emptyset, \forall u \in U$. The mapping T is continuous at x_0 if it is both $u.s.c.$ and $l.s.c.$ at x_0. Moreover, T is $u.s.c.$ $(l.s.c.)$ on X if T is $u.s.c.$ $(l.s.c.)$ at each point of X. We need the following basic definitions and results in the sequel.

Definition 1.2 [8] Let K be a non-empty subset of topological vector space X. A set-valued mapping $T: K \to 2^X$ is called a KKM mapping if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K, $Co\{x_1, x_2, \ldots, x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where Co denotes the convex hull.

The KKM -mappings were first considered by Knaster, Kuratowski and Mazurkiewicz (KKM) [11] in 1920, in order to guarantee the finite intersection property for values of the mapping.

Theorem 1.3 Let X and Y be two Hausdorff topological spaces and $T: X \longrightarrow 2^Y$ be a set-valued mapping with nonempty valued. Assume that T is closed valued and $u.s.c.$ mapping, then T has a closed graph.

2 MAIN RESULTS

The results of this section theorem can be viewed as an extension, improvement and repairmen of the Theorem 3.1 given in [8] by relaxing and weakening some assumptions.

Theorem 2.1 Let K be a nonempty convex subset of Hausdorff topological vector space X and $S: K \to 2^Y \backslash \emptyset$, where Y is a topological space. The set-valued mapping $F: K \times K \to 2^Y$, and single-valued mapping $g: K \to K$ satisfying in the following conditions.

(a) $F(g(x), x) \cap S(g(x)) \neq \emptyset, \forall x \in K$,

(b) $\{y \in K: F(x, y) \cap S(x) = \emptyset\}$ is convex, $\forall x \in K$,

(c) $\{x \in K: F(g(x), y) \cap S(g(x)) \neq \emptyset\}$ is closed, $\forall x \in K$,

(d) there exist compact convex set D and compact set M of K such that

$$\forall x \in K \backslash M, \exists y \in D, F(x, y) \cap S(x) = \emptyset. \tag{7}$$

Then there exists $x \in K$ such that the set

$$\{x \in K: F(x, y) \cap S(x) \neq \emptyset, \forall y \in K\}, \tag{8}$$

is nonempty and compact.

The next result is a direct consequence of Theorem 2.4 which is an improvement version of Corollary 2 in [1].

Corollary 2.2 Let K be a nonempty convex subset of Hausdorff topological vector space X and P is a nonempty subset of the topological space Y. If The mappings $F: K \times K \to Y$ and $g: K \to K$ satisfy the following conditions conditions.

(a) $F(g(x), x) \in P, \quad \forall x \in K,$

(b) $\{y \in K: F(x, y) \not\subseteq P\}$ is convex, $\forall x \in K,$

(c) $\{x \in K: F(g(x), y) \in P\}$ is closed, $\forall x \in K,$

(d) there exist compact convex set D and compact set M of K such that

$$\forall x \in K \backslash M, \exists y \in D, F(x, y) \not\subseteq P, \tag{9}$$

Then there exists $x \in K$ such that the set $\{x \in K: F(x, y) \in P, \forall y \in K\}$, is nonempty and compact.

3 APPLICATIONS

Let us recall the definition of the Nash equilibrium problem (NEP). There are N players, each player $v \in \{1, \ldots, N\}$ controls the variables $x^v \in R^{n_v}$. All players' strategies are collectively denoted by a vector $x = (x^1, x^2, \ldots, x^N)^T \in R^n$, where $n = n_1 + \cdots + n_N$. To emphasize the v th player's variables within the vector x, we sometimes write $x = (x^\nu, x^{-\nu})^T$, where $x^{-v} \in R^{n_{-v}}$ subsumes all the other players' variables.

Let $\theta_v: R^n \to R$ be the v th player's payoff (or loss or utility) function, and let $X^v \subseteq R^{n_v}$ be the strategy set of player v. Then, $x^* = (x^{*,1}, x^{*,2}, \ldots, x^{*,N})^T \in R^n$ is called a Nash equilibrium, or a solution of the NEP, if each block component $x^{*,v}$ is a solution of the optimization problem

$$\min \theta_v(x^\nu, x^{*,-\nu}) \text{ subject to } x^\nu \in X^v. \tag{10}$$

A generalized Nash equilibrium problem (GNEP) consists of p players. Each player v controls the decision variable $x^\nu \in C_v$, where C_v is a non-empty convex and closed subset of $R^{n\nu}$. We denote by $x = (x_1, \ldots, x_p) \in \prod_{v=1}^{p} C_v = C$ the vector formed by all these decision variables and by $x^{-\nu}$, we denote the strategy vector of all the players different from player ν. The set of all such vectors will be denoted by $C^{-\nu}$. We sometimes write $(x^\nu, x^{-\nu})$ instead of x in order to emphasize the ν-th player's variables within x. Note that this is still the vector $x = (x^1, \ldots, x^v, \ldots, x^p)$, and the notation $(x^\nu, x^{-\nu})$ does not mean that the block components of x are reordered in such a way that x^v becomes the first block. Each player v has an objective function $\theta_v: C \to R$ that depends on all player's strategies. Each player's strategy must belong to a set identified by the set-valued map $K_v: C^{-v} \rightrightarrows C_v$ in the sense that the strategy space of player v is $K_v(x^{-v})$, which depends on the rival player's strategies x^{-v}. Given the strategy x^{-v}, player v chooses a strategy x^v such that it solves the following optimization problem

$$\min \theta_v(x^\nu, x^{-\nu}) \text{ subject to } x^\nu \in K_v(x^{-v}), \tag{11}$$

for any given strategy vector x^{-v} of the rival players. The solution set of problem (11) is denoted by $\mathrm{Sol}_\nu(x^{-v})$. Thus, a generalized Nash equilibrium is a vector $x^\wedge$ such that $x^{\wedge v} \in \mathrm{Sol}_\nu(x^{\wedge -v})$, for any v. Associated to a GNEP, there is a function $f^{NI}: R^n \times R^n \to R$, defined by

$$f^{NI}(x, y) := \sum_{v=1}^{p} \{\theta_v(y^\nu, x^{-\nu}) - \theta_v(x^\nu, x^{-\nu})\}, \tag{12}$$

which is called Nikaido-Isoda function and was introduced in [6]. Additionally, we define the set-valued map $K: C \rightrightarrows C$ by

$$K(x) := \prod_{v=1}^{p} K_v(x^{-\nu}). \tag{13}$$

Definition 3.1 x^*, is a normalized Nash equilibrium of the GNEP, if $max_y f^{NI}(x^*, y) = 0$ holds, where $f^{NI}(x, y)$ denotes the Nikaido-Isoda function defined as (12).

The following theorem is a direct consequence of Corollary 2.2, which gives conditions of existence a normalized Nash equilibrium of the GNEP.

Theorem 3.2 Let K be a nonempty convex subset of Hausdorff topological vector space X and $P = (-\infty, 0]$ is a nonempty subset of the topological space Y. If The mappings $F: K \times K \to Y$ and $g: K \to K$ defined by

$$F(x,y) = f^{NI}(x,y) := \sum_{\nu=1}^{P} \{ \theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu})\}, \quad g(x) = x \tag{14}$$

and satisfy the following conditions.

(a) $\{y \in K: F(x,y) = f^{NI}(x,y) \not\subseteq P\}$ is convex, $\forall x \in K,$

(b) $\{x \in K: F(g(x),y) = f^{NI}(g(x),y) \in P\}$ is closed, $\forall x \in K,$

(c) there exist compact convex set D and compact set M of K such that

$$\forall x \in K \backslash M, \exists y \in D, F(x,y) = f^{NI}(x,y) \not\subseteq P, \tag{15}$$

Then there exists a normalized Nash equilibrium of the GNEP.

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