Some Results On Nikiforov Energy of Digraphs

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ABSTRACT

The energy of a graph G, $\mathcal{E}(G)$, is the sum of absolute values of the eigenvalues of its adjacency matrix. This concept was extended by Nikiforov to arbitrary complex matrices. The Nikiforov energy of a digraph D is defined as, $\mathcal{N}(D) = \sum_{i=1}^{n} \sigma_i$, where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of the adjacency matrix of D. In this paper, we show that for any digraph $D, \mathcal{N}(D) \geq \text{rank}(D)$ and the equality holds if and only if D is a disjoint union of directed cycles and directed paths. We prove that for a directed cycle C_n $\mathcal{N}(C_n) \leq n$. Finally, we characterize all digraphs with only one singular value.

KEYWORDS: Energy, Digraph, Singular value, Nikiforov enery.

1 INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. The energy of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of $A(G)$, see [5]. For a graph G, let $rank(G)$ denote the rank of the adjacency matrix of G.

For any matrix A, A^* is the *conjugate transpose* of A. The *singular values* of a matrix A, $\sigma_1 \geq \cdots \geq \sigma_n$, are defined as the square roots of the eigenvalues of A^*A. One can see that,

$$tr(A^*A) = \sum_{i=1}^{n} \sigma_i^2 = \sum_{1 \leq i,j \leq n} |a_{ij}|^2.$$

Let $D = (V, \mathcal{A})$ be a digraph without loops and multiple edges. Assume that the set of vertices of D is given by $\{1, \dots, n\}$. The adjacency matrix of D is defined as an $n \times n$ matrix A whose element a_{ij} is

$$a_{ij} = \begin{cases} 1 & \text{if (i, j) is an arc in } \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the singular values of D are the singular values of A. For any vertex $v \in V(D)$, let $N^+(v) = \{w: (v, w) \in \mathcal{A}\}$ and $d^+(v) = |N^+(v)|$ (the *outdegree* of v). Similarly, for any vertex $v \in V(D)$, let $N^-(v) = \{w: (w, v) \in \mathcal{A}\}$ and $d^-(v) = |N^-(v)|$ (the *indegree* of v). Let $\overleftarrow{C_n}$ and $\overrightarrow{C_n}$ denote the orientation of C_n whose directions are clockwise and counterclockwise, respectively. An *acyclic* digraph is a digraph having no directed cycle. Let A be the adjacency matrix of D, then A is nilpotent if and only if the directed graph D is acyclic.

Let $A \in M_{m\times n}(\mathbb{C})$. The *trace norm* of A, $\mathcal{N}(A)$ is defined as the sum of singular values of A. The concept of energy has been extended to digraphs as the trace norm of a digraph D, denoted by $\mathcal{N}(D)$, is the sum of singular values of D, [4]. Let D be a digraph of order n, size m, and with adjacency matrix A. Let $\sigma_1 \geq \cdots \geq \sigma_n$, be the singular values of A . Since $tr(A^*A) = \sum_{v\in V(G)} d^-(v) = m$, we have $\sum_{i=1}^{n} \sigma_i^2 = m$. This implies that $\sigma_n \leq \sqrt{\frac{m}{n}} \leq \sigma_1$.

The singular values of a directed cycle of order n, $\overleftarrow{C_n}$, are $\sigma_1 = \cdots = \sigma_n = 1$ and so $\mathcal{N}(\overleftarrow{C_n}) = n$, see [2, Example 2.3.]. The singular values of the directed path of order n, $\overleftarrow{P_n}$, are $\sigma_1 = \cdots = \sigma_{n-1} = 1$ and $\sigma_n = 0$. Hence $\mathcal{N}(\overleftarrow{P_n}) = n - 1$, see [1, Example 2.1.]

Lemma 1 [3, p. 238] *If $A \in M_n(\mathbb{C})$ is Hermitian, then $\lambda_{max}(A) \geq a_{ii} \geq \lambda_{min}(A)$, for all $i = 1, \dots, n$.*

2 MAIN RESULTS

In this section, we study the relation between rank and Nikiforov energy of a digraph. Furthermore, we investigate the Nikiforov energy of directed cycles.

Theorem 1. *Let D be a directed graph of order n. Then $\mathcal{N}(D) \geq rank(D)$ and the equality holds if and only if D is a disjoint union of directed cycles and directed paths.*

Proof. Let A be the adjacency matrix of D. Since A is real, $\mathrm{rank}(A^tA) = \mathrm{rank}(A)$. Let $\mathrm{rank}(A) = r$ and assume that $\sigma_1 \geq \cdots \geq \sigma_r$ are all non-zero singular values of D. If $f(x) = \sum_{i=0}^{n} a_i x^{n-i}$ is the characteristic polynomial of A^tA, then $a_i \in \mathbb{Z}$, for $i = 0, \dots, n$ and $a_r = (-1)^r \sigma_1^2 \cdots \sigma_r^2 \neq 0$. Thus $\sigma_1^2 \cdots \sigma_r^2 \geq 1$. Then arithmetic-geometric inequality implies that

$$\frac{\sigma_1 + \cdots + \sigma_r}{r} \geq \sqrt[r]{\sigma_1 \cdots \sigma_r} = \sqrt[2r]{\sigma_1^2 \cdots \sigma_r^2} \geq 1. \tag{1}$$

Thus $\mathcal{N}(D) \geq \mathrm{rank}(D)$. In order to prove the last assertion, first assume that $\overleftarrow{C_n}$ is a directed cycle of order n. Since the adjacency matrix of $\overleftarrow{C_n}$, is non-singular, $\mathrm{r}ank(\overleftarrow{C_n}) = n$. Furthermore $\mathcal{N}(\overleftarrow{C_n}) = n$. Next, assume that $\overleftarrow{P_n}$ is a directed path of order n. Then $\mathrm{r}ank(\overleftarrow{P_n}) = n - 1$. In particular, by Theorem 3, $\mathcal{N}(\overleftarrow{P_n}) = n - 1$. Now, let D be a disjoint union of c directed cycles and p directed paths, we find that $\mathcal{N}(D) = \mathrm{r}ank(D) = n - p$.

Conversely, suppose that $\mathcal{N}(D) = \mathrm{rank}(D)$, so the equality holds in (1), that is $\sigma_1 = \cdots = \sigma_r = 1$. By Lemma 1, $\sigma_1^2 \geq (A^tA)_{ii}$. Since $(A^tA)_{ii} = d^-(v_i)$, we find that for each i, $d^-(v_i) \leq 1$. By a similar method, we obtain $d^+(v_i) \leq 1$. This implies that each component of D is either a directed cycle or a directed path.

Theorem 2. *Let $n \geq 3$ be a positive integer. Then for any orientation of C_n, $\mathcal{N}(C_n) < n$, except for clockwise and counterclockwise.*

Proof. Consider C_n with an arbitrary orientation, which is not directed cycles $\overleftarrow{C_n}$ and $\overrightarrow{C_n}$. Since this directed graph is acyclic, then A is nilpotent, where A is the adjacency matrix of the directed C_n. Now, if $\sigma_1 \geq \cdots \geq \sigma_n$ are singular values of the directed C_n, then $\sigma_n = 0$. Thus the following holds:

$$\mathcal{N}(C_n) = \sum_{i=1}^{n-1} \sigma_i \leq (n-1)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \sigma_i^2\right)^{\frac{1}{2}} = \sqrt{n(n-1)} < n,$$

the proof is complete.

Theorem 3. *Let D be a digraph with only one singular value. Then D is a union of isolated vertices or a disjoint union of directed cycles.*

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