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Multi-super-stability of ternary antiderivation in ternary Banach algebras

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Abstract

In this study, we introduce the notion of ternary antiderivation on ternary Banach algebras and investigate the multi-super-stability of ternary antiderivation in ternary Banach algebras, associated with functional inequalities.

Keywords: Hyers-Ulam stability; multi-super-stability; fixed point method; ternary antiderivation, ternary Banach algebra; additive functional inequality.

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1 Introduction

First, we recall a fundamental result in fixed point theory.

Assume Banach algebras $\mathscr{X}$ and $\mathscr{X}''$. Suppose $(\mathscr{X}', \Delta)$ is a probability measure space and suppose $(\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ and $(\mathscr{X}'', \mathfrak{B}_{\mathscr{X}''})$ are Borel measurable spaces. Then a map $\mathscr{J} : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}''$ is a operator if $\{\mathfrak{P} : \mathscr{J}(\mathfrak{P}, \alpha) \in \nu\} \in \Delta$ for each α in $\mathscr{X}$ and $\nu \in \mathfrak{B}_{\mathscr{X}''}$. Now, we are going to propose vector valued generalized metric spaces. Assume $\mho = (\mho_1, \ldots, \mho_m)$ and $\Omega = (\Omega_1, \ldots, \Omega_m), m \in \mathbb{N}$. Then we have

$$\mho \preceq \Omega \Longleftrightarrow \mho_\imath \leq \Omega_\imath, \quad \imath = 1, \cdots, m;$$

and also

$$\mho \to 0 \Longleftrightarrow \mho_\imath \to 0, \quad \imath = 1, \cdots, m.$$

Definition 1.1 ([1])**.** Let $\nabla \neq \emptyset$ is a set and $d : \nabla^2 \to [0, +\infty]^m, m \in \mathbb{N}$, is a given mapping. If the following conditions are satisfied, then we say d is a generalized metric on ∇:

▷ for each $(g, g') \in \nabla \times \nabla$, we get

$$d(g, g') = \underbrace{(0, \cdots, 0)}_{m} \Longleftrightarrow g = g';$$

[1]speaker

▷ for each $(g, g') \in \nabla \times \nabla$, we get

$$d(g', g) = d(g, g') \iff g = g';$$

▷ for each $g, g', g'' \in \nabla$, we get

$$d(g, g'') + d(g'', g') \succeq d(g', g).$$

Theorem 1.2 ([1]). *Assume the following assumptions:*

▷ $d : \nabla^2 \to [0, +\infty]^m, m \in \mathbb{N}$, *and* (∇, d) *is a complete generalized metric space.*

▷ $\mathscr{L} : \nabla \to \nabla$ *is a strictly contractive mapping with Lipschitz constant* $\mathcal{Z} < 1$.

Then for each $g \in \nabla$, *either*

$$d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) = \overbrace{(+\infty, \cdots, +\infty)}^{m}$$

for each $n \in \mathbb{N} \cup \{0\}$ *or there is a* $n_0 \in \mathbb{N}$ *such that*

▷ $d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) \preceq \overbrace{(+\infty, \cdots, +\infty)}^{m}, \qquad \forall n \geq n_0;$

▷ *the sequence* $\{\mathscr{L}^n g\}$ *converges to a fixed point* $(g')^*$ *of* $\mathscr{L}$;

▷ $(g')^*$ *is the unique fixed point of* $\mathscr{L}$ *in the set* $\complement = \{g' \in \nabla \mid d(\mathscr{L}^{n_0} g, g') \preceq \overbrace{(+\infty, \cdots, +\infty)}^{m}\}$;

▷ $d(g', (g')^*) \preceq \frac{1}{1-\mathcal{Z}} d(g', \mathscr{L} g')$ *for each* $g' \in \complement$.

Now, by introducing some special functions, we present the concept of multi-stability.

Consider the following special functions [1].

•The Gauss hypergeometric series :

$$\varphi_1^{\circledS}(X) := {}_2\mathbb{F}_1(\alpha, \mathsf{B}; \mathsf{T}; X) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\mathsf{B})_n}{(\mathsf{T})_n} \frac{X^n}{n!},$$

where $\alpha, \mathsf{B}, \mathsf{T}, X \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, and $|X| < 1$.

• The Clausen hypergeometric series :

$$\varphi_2^{\circledS}(X) := {}_p\mathbb{F}_q((\alpha); (\mathsf{T}); X) = {}_p\mathbb{F}_q\left(\begin{smallmatrix}\alpha_1, \cdots, \alpha_p \\ \mathsf{T}_1, \cdots, \mathsf{T}_q\end{smallmatrix}; X\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\mathsf{T}_1)_k \cdots (\mathsf{T}_q)_k} \frac{X^n}{n!}, \tag{1}$$

where $p, n, q \in \mathbb{N} \cup \{0\}$ and $\alpha_n, X, \mathsf{T}_n \in \mathbb{C}$.

• The Wright generalized hypergeometric function. Assume $\Xi := -\sum_{k=1}^{q} b_k + \sum_{j=1}^{p} a_j$, $\sigma := -\prod_{k=1}^{q} |b_k|^{-b_k} + \prod_{j=1}^{p} |a_j|^{-a_j}$, and $\chi := -\sum_{j=1}^{p} \kappa_j + \sum_{k=1}^{q} \vartheta_k + \dfrac{p-q}{2}$, where $\kappa_j, \vartheta_k \in \mathbb{C}$, $k, j \in \mathbb{N}$, $p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

Therefore Wright generalized hypergeometric series is given by

$$\varphi_3^{\circledS}(X) = {}_p\mathbb{W}_q\left(\begin{smallmatrix}(\kappa_p, a_p)_{1,p} \\ (\vartheta_q, b_q)_{1,q}\end{smallmatrix}; X\right) = \sum_{s=0}^{\infty} \frac{\left\{\prod_{j=1}^{p} \Gamma(\kappa_j + a_j s)\right\}}{\left\{\prod_{k=1}^{q} \Gamma(\vartheta_k + b_k s)\right\}} \frac{X^s}{s!}, \tag{2}$$

where $j, s, k \in \mathbb{N}, X \in \mathbb{C}$, $\Xi > -1, \kappa_j, \vartheta_k \in \mathbb{C}$, $p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

• The Wright function :

$$\varphi_4^{\circledS}(X) := \mathbb{K}(\vartheta, b, X) = {}_0\mathbb{W}_1\left(\begin{smallmatrix}- \\ (b, \vartheta)\end{smallmatrix}; X\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\vartheta + bk)} \frac{X^k}{k!}, \tag{3}$$

where $X, \vartheta \in \mathbb{C}$, and $b \in \mathbb{R}$.

• The Wright generalized Bessel function (Bessel–Maitland function) :

$$\varphi_5^{Ⓢ}(X) := \mathbb{J}(\kappa, a, X) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\kappa + 1 + ak)} \frac{(-X)^k}{k!} = {}_0\mathbb{W}_1\left({}^{-}_{(\kappa+1,b)}; -X\right),$$

where $\kappa, X \in \mathbb{C}$, and $a \in \mathbb{R}$.

•The shifted Wright generalized hypergeometric series :

$$\varphi_6^{Ⓢ}(X) = {}_p\mathbb{B}_q\left({}^{(\kappa_p, a_p; \vartheta_p, b_p)_{1,p}}_{(\widehat{\kappa}_p, c_p; \widehat{\vartheta}_p, d_p)_{1,q}}; X\right) = \sum_{k=0}^{\infty} \frac{\left\{\prod_{m=1}^{p} b(\kappa_m + a_m k; \vartheta_m + b_m k)\right\}}{\left\{\prod_{n=1}^{q} b(\widehat{\kappa}_n + c_n k; \widehat{\vartheta}_n + d_n k)\right\}} \frac{X^k}{k!},$$

where $m, n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}, \kappa_m, \vartheta_m, \widehat{\kappa}_n, \widehat{\vartheta}_n, X \in \mathbb{C},\ p, q \in \mathbb{N} \cup \{0\}$, and $a_m, b_m, c_n, d_n \in \mathbb{R}_+$.

Now, we define the Wright generalized hypergeometric series as follows

$$\varphi_7^{Ⓢ}(X) := [{}_p\mathbb{W}_q]^n(X) = \sum_{s=0}^{n} \frac{\left\{\prod_{j=1}^{p} \Gamma(\kappa_j + a_j s)\right\}}{\left\{\prod_{k=1}^{q} \Gamma(\vartheta_k + b_k s)\right\}} \frac{X^s}{s!},$$

where $X, \kappa_j, \vartheta_k \in \mathbb{C}, s, j, k, q, p \in \mathbb{N}$, and $a_j, b_k \in \mathbb{R}_+$.

Let

$$\mathrm{diag}[\rho_1, \cdots, \rho_n]_{n\times n} = \begin{bmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \rho_n \end{bmatrix}_{n\times n}.$$

Note that $\rho := \mathrm{diag}[\rho_1, \cdots, \rho_n] \preceq \varrho := \mathrm{diag}[\varrho_1, \cdots, \varrho_n]$ if $\rho_i \leq \varrho_i$ for each $1 \leq i \leq n$.

Assume the following matrix valued control function given by

$$\mathfrak{W}[X] = \mathrm{diag}\left[\varphi_1^{Ⓢ}(X), \cdots, \varphi_n^{Ⓢ}(X)\right]_{n\times n}.$$

Assume a mapping Υ from a vector space κ to normed linear space ϑ has Hyers-Ulam-Rassias stability. If we replace the control function of Hyers-Ulam-Rassias stability with $\mathfrak{W}[X]$, we say Υ has the multi-stability property.

In this paper, we investigate the multi-stability problem of ternary antiderivations associated to the following functional inequality in complex ternary Banach algebras through the fixed point method.

$$\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|\mathscr{J}_1(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_1(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma)-\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)\Big\| \\
&\qquad ,\cdots,\ \Big\|\mathscr{J}_n(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_n(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_n(\mathfrak{P},\alpha+\beta-\gamma)-\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)\Big\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ \Big\|\theta_1(\mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_1(\mathfrak{P},\beta))\Big\| \\
&\qquad\qquad + \Big\|\theta_1'(\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{J}_1(\mathfrak{P},\gamma))\Big\|\ ,\cdots, \\
&\qquad\quad \Big\|\theta_n(\mathscr{J}_n(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_n(\mathfrak{P},\beta))\Big\| \\
&\qquad\qquad + \Big\|\theta_n'(\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{J}_n(\mathfrak{P},\gamma))\Big\|\ \Big],
\end{aligned} \tag{4}$$

for each $\alpha,\beta,\gamma\in\mathscr{X}$, $\mathfrak{P}\in\mathscr{X}'$, where $\theta_1,\cdots,\theta_n$ and $\theta_1',\cdots,\theta_n'$ are fixed nonzero complex numbers with $|\theta_1|+|\theta_1'|<2,\cdots,|\theta_n|+|\theta_n'|<2$.

2 investigating the multi-stability and super-multi-stability associated to (4)

In this section, we investigate the concept of ternary antiderivation on ternary Banach algebras and introduce the super-multi-stability of ternary antiderivation in ternary Banach algebras, associated to the (4). For more details, see [2, 3, 4, 5, 6, 7].

Throughout this section, let $\mathscr{X}$ be a ternary Banach algebra. and $\theta:=(\theta_1,\cdots,\theta_n)$ and $\theta':=(\theta_1',\cdots,\theta_n')$ are fixed nonzero complex numbers with $|\theta_1|+|\theta_1'|<2,\cdots,|\theta_n|+|\theta_n'|<2$.

A ternary Banach algebra is a complex Banach space $\mathscr{X}$, endowed with a ternary product $(\alpha,\beta,\gamma)\to[\alpha,\beta,\gamma]$ of $\mathscr{X}\times\mathscr{X}\times\mathscr{X}$ into $\mathscr{X}$, which $\mathbb{C}$-linear in the each variables, and associative in the sense that $[\alpha,\beta,[\gamma,\omega,\upsilon]]=[\alpha,[\omega,\gamma,\beta],\upsilon]=[[\alpha,\beta,\gamma],\omega,\upsilon]$, and satisfies $\|[\alpha,\beta,\gamma]\|\le\|\alpha\|\cdot\|\beta\|\cdot\|\gamma\|$ for each $\alpha,\beta,\gamma,\omega,\upsilon\in\mathscr{X}$.

If a ternary Banach algebra $(\mathscr{X},[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $\varrho\in\mathscr{X}$ such that $\alpha=[\alpha,\varrho,\varrho]=[\varrho,\varrho,\alpha]$ for each $\alpha\in\mathscr{X}$, then it is routine to verify that $\mathscr{X}$, endowed with $\alpha\circ\beta:=[\alpha,\varrho,\beta]$ and $\alpha^*:=[\varrho,\alpha,\varrho]$, is a unital algebra. Conversely, if $(\mathscr{X},\circ)$ is a unital algebra, then $[\alpha,\beta,\gamma]:=\alpha\circ\beta^*\circ\gamma$ makes $\mathscr{X}$ into a ternary Banach algebra. If a ternary Banach algebra $(\mathscr{X},[\cdot,\cdot,\cdot])$ has a unit, i.e., an element $\varrho\in\mathscr{X}$ s.t. $\alpha=[\alpha,\varrho,\varrho]=[\varrho,\varrho,\alpha]$ for each $\alpha\in\mathscr{X}$, then $\mathscr{X}$ with the binary product $\alpha\circ\beta:=[\alpha,\varrho,\beta]$, is a usual Banach algebra.

A $\mathbb{C}$-linear mapping $\varpi:\mathscr{X}'\times\mathscr{X}\to\mathscr{X}$ is called a ternary homomorphism if $\varpi(\mathfrak{P},[\alpha,\beta,\gamma])=[\varpi(\mathfrak{P},\alpha),\varpi(\mathfrak{P},\beta),\varpi(\mathfrak{P},\gamma)]$ for each $\alpha,\beta,\gamma\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$. A $\mathbb{C}$-linear mapping $\varpi':\mathscr{X}'\times\mathscr{X}\to\mathscr{X}$ is called a ternary derivation if $\varpi'(\mathfrak{P},[\alpha,\beta,\gamma])=[\varpi'(\mathfrak{P},\alpha),\beta,\gamma]+[\alpha,\varpi'(\mathfrak{P},\beta),\gamma]+[\alpha,\beta,\varpi'(\mathfrak{P},\gamma)]$ for each $\alpha,\beta,\gamma\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$.

2.1 Multi-stability of (θ,θ')-functional inequality (4)

In this subsection, we propose the multi-stability of the additive (θ,θ')-functional inequality (4) through the fixed point method.

Lemma 2.1. *Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying (4), for each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are additive.*

Proof. Let $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, satisfies (4). Letting $\alpha = \beta = \gamma = 0$ in (4), we obtain $2\|\mathscr{J}_i(\mathfrak{P}, 0)\| \leq (|\theta_i| + |\theta'_i|)\|\mathscr{J}_i(\mathfrak{P}, 0)\|$, for $i = 1, \cdots, n$ and therefore $\mathscr{J}_i(\mathfrak{P}, 0) = 0$, since $|\theta_i| + |\theta'_i| < 2$. Putting $\gamma = \alpha$ in (4), we get $\|\mathscr{J}_i(\mathfrak{P}, 2\alpha + \beta) - \mathscr{J}_i(\mathfrak{P}, 2\alpha) - \mathscr{J}_i(\mathfrak{P}, \beta)\| \leq 0$ and so $\mathscr{J}_i(\mathfrak{P}, 2\alpha + \beta) = \mathscr{J}_i(\mathfrak{P}, 2\alpha) + \mathscr{J}_i(\mathfrak{P}, \beta)$ for each $\alpha, \beta \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Therefore $\mathscr{J}_i, (i = 1, \cdots, n)$, are additive. □

Theorem 2.2. *Suppose $\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}, \cdots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n} : \mathscr{X}^3 \to [0, \infty)$ are functions s.t. there exist $(\mathcal{T}_1, \cdots, \mathcal{T}_n) \prec \overbrace{(1, \cdots, 1)}^{n}$ with*

$$\begin{aligned} &diag\Big[\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\alpha,\beta,\gamma), \cdots, \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\alpha,\beta,\gamma) \Big] \\ &\preceq diag\Big[\frac{\mathcal{T}_1}{2} \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(2\alpha,2\beta,2\gamma), \cdots, \frac{\mathcal{T}_n}{2} \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(2\alpha,2\beta,2\gamma) \Big], \end{aligned} \tag{5}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying

$$\begin{aligned} &diag\Big[\Big\|\mathscr{J}_1(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\Big\| \\ &\quad , \cdots, \Big\|\mathscr{J}_n(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\Big\| \Big] \\ &\preceq diag\Big[\Big\|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\Big\| \\ &\quad + \Big\|\theta'_1(\mathscr{J}_1(\mathfrak{P}, \alpha-z) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\Big\| + \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\alpha,\beta,\gamma) \end{aligned} \tag{6}$$

$$\begin{aligned} &\quad , \cdots, \Big\|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\Big\| \\ &\quad + \Big\|\theta'_n(\mathscr{J}_n(\mathfrak{P}, \alpha-z) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\Big\| + \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\alpha,\beta,\gamma) \Big], \end{aligned} \tag{7}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Then there exist unique additive mappings $\mathscr{J}'_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, s.t.

$$\begin{aligned} &diag\Big[\|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}'_n(\mathfrak{P}, \alpha)\| \Big] \\ &\preceq diag\Big[\frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)} \underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\Big(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\Big), \cdots, \frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)} \underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\Big(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\Big) \Big], \end{aligned} \tag{8}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Proof. Letting $\alpha=\beta=\gamma=0$ in (6), we have

$$\begin{aligned}&\operatorname{diag}\Big[\ 2\|\mathscr{J}_1(\mathfrak{P},0)\|,\cdots,2\|\mathscr{J}_n(\mathfrak{P},0)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ (|\theta_1|+|\theta_1'|)\|\mathscr{J}_1(\mathfrak{P},0)\|+\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(0,0,0),\cdots,\\ &\qquad (|\theta_n|+|\theta_n'|)\|\mathscr{J}_n(\mathfrak{P},0)\|+\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(0,0,0)\ \Big]\end{aligned}$$

and thus $\mathscr{J}_i(\mathfrak{P},0)=0,(i=1,\cdots,n)$, since $|\theta_1|+|\theta_1'|,\cdots,|\theta_n|+|\theta_n'|<2$ and by (5),

$$\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(0,0,0),\cdots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(0,0,0)=0.$$

Putting $\alpha=\gamma=\frac{\tau}{2}$ and $\beta=\tau$ in (6), we obtain

$$\begin{aligned}&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},2\tau)-2\mathscr{J}_1(\mathfrak{P},\tau)\|,\cdots,\|\mathscr{J}_n(\mathfrak{P},2\tau)-2\mathscr{J}_n(\mathfrak{P},\tau)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\tau}{2},\tau,\frac{\tau}{2}\right),\cdots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\tau}{2},\tau,\frac{\tau}{2}\right)\ \Big],\end{aligned}\tag{9}$$

for each $\tau\in\mathscr{X}$, $\mathfrak{P}\in\mathscr{X}'$.

Throughout this subsection, let $\hbar:=(\hbar_1,\cdots,\hbar_n)$ and $\hbar':=(\hbar_1',\cdots,\hbar_n')$.

Now, consider the set

$$\nabla=\Big\{\hbar:\overbrace{(\mathscr{X}'\times\mathscr{X})\times\cdots\times(\mathscr{X}'\times\mathscr{X})}^{n}\to\overbrace{\mathscr{X}\times\cdots\times\mathscr{X}}^{n}:\hbar(\mathfrak{P},0)=\overbrace{(0,\cdots,0)}^{n}\Big\}$$

and the mapping d defined on $\nabla\times\nabla$ by

$$\begin{aligned}d(\hbar,\hbar')=\inf\Big\{&(\mu_1,\cdots,\mu_n)\in\mathbb{R}_+^n:\\ &\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P},\alpha)-\hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\hbar_n(\mathfrak{P},\alpha)-\hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \mu_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\mu_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],\forall\alpha\in\mathscr{X},\ \mathfrak{P}\in\mathscr{X}'\Big\},\end{aligned}$$

where as usual, $\inf\emptyset=(+\infty,\cdots,+\infty)$. d is a complete generalized metric on ∇.

Now, let us consider the linear mapping $\mathscr{L}:=(\mathscr{L}_1,\cdots,\mathscr{L}_n):\nabla\to\nabla$ such that $\mathscr{L}_i\hbar_i(\mathfrak{P},\alpha):=2\hbar_i\left(\mathfrak{P},\frac{\alpha}{2}\right)$ for each $\alpha\in\mathscr{X}$, $\mathfrak{P}\in\mathscr{X}'$.

Indeed, suppose $\hbar,\hbar'\in\nabla$ are given s.t. $d(\hbar,\hbar')=(\varepsilon_1,\cdots,\varepsilon_n)$, then

$$\begin{aligned}&\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P},\alpha)-\hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\hbar_n(\mathfrak{P},\alpha)-\hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \varepsilon_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\varepsilon_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Hence

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{L}_1\hbar_1(\mathfrak{P},\alpha) - \mathscr{L}_1\hbar_1'(\mathfrak{P},\alpha)\|, \cdots, \|\mathscr{L}_n\hbar_n(\mathfrak{P},\alpha) - \mathscr{L}_n\hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\
&= \operatorname{diag}\Big[\ \Big\|2\hbar_1\left(\mathfrak{P},\frac{\alpha}{2}\right) - 2\hbar_1'\left(\mathfrak{P},\frac{\alpha}{2}\right)\Big\|, \cdots, \Big\|2\hbar_n\left(\mathfrak{P},\frac{\alpha}{2}\right) - 2\hbar_n'\left(\mathfrak{P},\frac{\alpha}{2}\right)\Big\|\ \Big]\\
&\preceq \operatorname{diag}\Big[\ 2\varepsilon_1 \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{4},\frac{\alpha}{2},\frac{\alpha}{4}\right), \cdots, 2\varepsilon_n \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{4},\frac{\alpha}{2},\frac{\alpha}{4}\right)\ \Big]\\
&\preceq \operatorname{diag}\Big[\ \mathcal{T}_1\varepsilon_1 \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right), \cdots, \mathcal{T}_n\varepsilon_n \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, that is $d(\hbar,\hbar') = (\varepsilon_1,\cdots,\varepsilon_n)$ implies that $d(\mathscr{L}\hbar(\mathfrak{P},\alpha), \mathscr{L}\hbar'(\mathfrak{P},\alpha)) \preceq (\mathcal{T}_1\varepsilon_1,\cdots,\mathcal{T}_n\varepsilon_n)$. This means that $d(\mathscr{L}\hbar(\mathfrak{P},\alpha), \mathscr{L}\hbar'(\mathfrak{P},\alpha)) \preceq (\mathcal{T}_1,\cdots,\mathcal{T}_n)d(\hbar,\hbar')$ for each $\hbar,\hbar' \in \nabla$.

Next, from (9), we get

$$\begin{aligned}
&\operatorname{diag}\Big[\ \Big\|\mathscr{J}_1(\mathfrak{P},\alpha) - 2\mathscr{J}_1(\mathfrak{P},\frac{\alpha}{2})\Big\|, \cdots, \Big\|\mathscr{J}_n(\mathfrak{P},\alpha) - 2\mathscr{J}_n(\mathfrak{P},\frac{\alpha}{2})\Big\|\ \Big]\\
&\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{4},\frac{\alpha}{2},\frac{\alpha}{4})\right), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{4},\frac{\alpha}{2},\frac{\alpha}{4})\right)\ \Big]\\
&\preceq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2} \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right), \cdots, \frac{\mathcal{T}_n}{2} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, it follows that $d(\mathscr{J},\mathscr{L}\mathscr{J}) \preceq (\frac{\mathcal{T}_1}{2},\cdots,\frac{\mathcal{T}_n}{2})$,in which $\mathscr{J} := (\mathscr{J}_1,\cdots,\mathscr{J}_n)$.

by the fixed point alternative we conclude the existence of unique fixed points of $\mathscr{L}_i, (i = 1,\cdots,n)$, that are, the existence of a mapping $\mathscr{J}_i' : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$ such that $\mathscr{J}_i'(\mathfrak{P},\alpha) = 2\mathscr{J}_i'\left(\mathfrak{P},\frac{\alpha}{2}\right), \quad i = 1,\cdots,n$ with the following property: there exist $(\mu_1,\cdots,\mu_n) \in (0,\infty)^n$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\alpha) - \mathscr{J}_1'(\mathfrak{P},\alpha)\|, \cdots, \|\mathscr{J}_1(\mathfrak{P},\alpha) - \mathscr{J}_1'(\mathfrak{P},\alpha)\|\\
&\preceq \operatorname{diag}\Big[\ \mu_1 \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right), \cdots, \mu_n \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Since $\lim_{m\to\infty} d(\mathscr{L}^m\mathscr{J},\mathscr{J}') = \underbrace{(0,\cdots,0)}_{n}$,

$$\lim_{m\to\infty} 2^m \mathscr{J}\left(\mathfrak{P},\frac{\alpha}{2^m}\right) = \mathscr{J}'(\mathfrak{P},\alpha)$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, and $\mathscr{J} := (\mathscr{J}_1,\cdots,\mathscr{J}_n)$, $\mathscr{J}' := (\mathscr{J}_1',\cdots,\mathscr{J}_n')$.

Also, $d(\mathscr{J},\mathscr{J}') \preceq \overbrace{(\frac{1}{1-\mathcal{T}_1},\cdots,\frac{1}{1-\mathcal{T}_1})}^{n} d(\mathscr{J},\mathscr{L}\mathscr{J})$ which implies

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\alpha) - \mathscr{J}_1'(\mathfrak{P},\alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P},\alpha) - \mathscr{J}_n'(\mathfrak{P},\alpha)\|\ \Big]\\
&\preceq \frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)} \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right), \cdots, \frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. It follows from (5) and (6) that

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1'(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_1'(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_1'(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)\|\ ,\cdots,\\
&\qquad\|\mathscr{J}_n'(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_n'(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_n'(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)\|\Big]\\
&=\operatorname{diag}\Big[\ \lim_{m\to\infty}2^m\Big\|\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\gamma}{2^n}\right)\\
&\qquad-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)\Big\|\ ,\cdots,\\
&\qquad\lim_{m\to\infty}2^m\Big\|\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m}\right)\\
&\qquad-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)\Big\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \lim_{m\to\infty}2^m\left\|\theta_1\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta}{2^m}\right)\right)\right\|\\
&\qquad+\lim_{m\to\infty}2^m\left\|\theta_1'\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\gamma}{2^m}\right)\right)\right\|\\
&\qquad+\lim_{m\to\infty}2^m\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\mathfrak{P},\frac{\alpha}{2^m},\frac{\beta}{2^m},\frac{\gamma}{2^m}\right)\ ,\cdots,\\
&\qquad\lim_{m\to\infty}2^m\left\|\theta_n\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta}{2^m}\right)\right)\right\|\\
&\qquad+\lim_{m\to\infty}2^m\left\|\theta_n'\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\gamma}{2^m}\right)\right)\right\|\\
&\qquad+\lim_{m\to\infty}2^m\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\mathfrak{P},\frac{\alpha}{2^m},\frac{\beta}{2^m},\frac{\gamma}{2^m}\right)\ \Big]\\
&=\operatorname{diag}\Big[\ \|\theta_1(\mathscr{J}_1'(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_1'(\mathfrak{P},\beta))\|\\
&\qquad+\|\theta_1'(\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_1'(\mathfrak{P},\alpha)-\mathscr{J}_1'(\mathfrak{P},\gamma))\|,\cdots,\\
&\qquad\|\theta_n(\mathscr{J}_n'(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_n'(\mathfrak{P},\beta))\|\Big]\\
&\qquad+\|\theta_n'(\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_n'(\mathfrak{P},\alpha)-\mathscr{J}_n'(\mathfrak{P},\gamma))\|\ \Big],
\end{aligned}$$

for each $\alpha,\beta,\gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Therefore, by Lemma 2.1, the mappings $\mathscr{J}_i'$, $(i = 1,\cdots,n)$, are additive. □

2.2 Ternary antiderivations in ternary algebras

In this subsection, we propose the concept of ternary antiderivation in ternary Banach algebras and investigate the super-multi-stability of ternary antiderivations associated to (4) in ternary Banach algebras.

Definition 2.3. Let $\mathscr{X}$ be a ternary Banach algebra. $\mathbb{C}$-linear mappings $\mathscr{G}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$, $(i = 1,\cdots,n)$, are called ternary antiderivations if they satisfy

$$[\mathscr{G}_i(\mathfrak{P},\alpha),\mathscr{G}_i(\mathfrak{P},\beta),\mathscr{G}_i(\mathfrak{P},\gamma)] = \mathscr{G}_i[\mathfrak{P},\mathscr{G}_i(\mathfrak{P},\alpha),\beta,\gamma] + \mathscr{G}_i[\mathfrak{P},\alpha,\mathscr{G}_i(\mathfrak{P},\beta),\gamma] + \mathscr{G}_i[\mathfrak{P},\alpha,\beta,\mathscr{G}_i(\mathfrak{P},\gamma)]$$

for each $\alpha,\beta,\gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$ and $i = 1,\cdots,n$.

Lemma 2.4. *Assume $\mathscr{X}$ be a complex Banach algebra and assume $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, be additive mappings s.t. $\mathscr{J}_i(\mathfrak{P}, \mathfrak{J}\alpha) = \mathfrak{J}\mathscr{J}_i(\mathfrak{P}, \alpha), (i = 1, \cdots, n)$, for each $\mathfrak{J} \in \mathbb{T}^1 := \{\kappa \in \mathbb{C} : |\kappa| = 1\}$ and each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then $\mathscr{J}_i, (i = 1, \cdots, n)$, are $\mathbb{C}$-linear.*

Theorem 2.5. *Suppose $\underbrace{\varphi_{j_1}}_{1\le j_1\le n}, \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n} : \mathscr{X}^3 \to [0, \infty)$ be functions. If there exist $(\mathcal{T}_1, \cdots, \mathcal{T}_n) < (1, \cdots, 1)$ with satisfying*

$$
\begin{aligned}
&diag\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right)\ \Big] \\
&\preceq diag\Big[\ \frac{\mathcal{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathcal{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\le j_n\le n}(2\alpha, 2\beta, 2\gamma)\ \Big], \qquad (10)
\end{aligned}
$$

for ech $\mathfrak{J} \in \mathbb{T}^1$ and all $\alpha, \beta, \gamma \in \mathscr{X}$. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings s.t.

$$
\begin{aligned}
&diag\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots, \\
&\qquad \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\|\ \Big] \\
&\preceq diag\Big[\ \|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha, \beta, \gamma), \cdots, \\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha, \beta, \gamma)\ \Big], \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
&diag\Big[\ \|[\mathscr{J}_1(\mathfrak{P}, \alpha), \mathscr{J}_1(\mathfrak{P}, \beta), \mathscr{J}_1(\mathfrak{P}, \gamma)] - \mathscr{J}_1[\mathfrak{P}, \mathscr{J}_1(\mathfrak{P}, \alpha), \beta, \gamma] \qquad (12) \\
&\qquad + \mathscr{J}_1[\mathfrak{P}, \alpha, \mathscr{J}_1(\mathfrak{P}, \beta), \gamma] + \mathscr{J}_1[\mathfrak{P}, \alpha, \beta, \mathscr{J}_1(\mathfrak{P}, \gamma)]\|, \cdots, \\
&\qquad \|[\mathscr{J}_n(\mathfrak{P}, \alpha), \mathscr{J}_n(\mathfrak{P}, \beta), \mathscr{J}_n(\mathfrak{P}, \gamma)] - \mathscr{J}_n[\mathfrak{P}, \mathscr{J}_n(\mathfrak{P}, \alpha), \beta, \gamma] \\
&\qquad + \mathscr{J}_n[\mathfrak{P}, \alpha, \mathscr{J}_n(\mathfrak{P}, \beta), \gamma] + \mathscr{J}_n[\mathfrak{P}, \alpha, \beta, \mathscr{J}_n(\mathfrak{P}, \gamma)]\|\ \Big] \\
&\preceq diag\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha, \beta, \gamma), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha, \beta, \gamma)\ \Big],
\end{aligned}
$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. If $\mathscr{J}_i, (i = 1, \cdots, n)$, are continuous and $\mathscr{J}_i(\mathfrak{P}, 2\alpha) = 2\mathscr{J}_i(\mathfrak{P}, \alpha)$ for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are ternary antiderivations.

Proof. Let $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, satisfie (11).

Letting $\mathfrak{J} = 1$ and $\alpha = \beta = \gamma = 0$ in (11), we get

$$\begin{aligned}&\operatorname{diag}\Big[\ 2\|\mathscr{J}_1(\mathfrak{P},0)\|,\cdots,2\|\mathscr{J}_n(\mathfrak{P},0)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ (|\theta_1|+|\theta_1'|)\|\mathscr{J}_1(\mathfrak{P},0)\| + \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(0,0,0)\ ,\cdots,\\ &\qquad (|\theta_n|+|\theta_n'|)\|\mathscr{J}_n(\mathfrak{P},0)\| + \underbrace{\varphi_{j_n}}_{1\le j_1\le n}(0,0,0)\ \Big],\end{aligned}$$

and thus $\mathscr{J}_i(\mathfrak{P},0) = 0, (i = 1,\cdots,n)$, since $|\theta_1| + |\theta_1'|,\cdots,|\theta_n| + |\theta_n'| < 2$ and by (10),

$$\underbrace{\varphi_{j_1}}_{1\le j_1\le n}(0,0,0),\cdots,\ \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(0,0,0) = 0.$$

Putting $\alpha = \gamma = \frac{\tau}{2}$ and $\gamma = \tau$ in (11), we obtain

$$\begin{aligned}&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},2\mathfrak{J}\tau) - 2\mathfrak{J}\mathscr{J}_1(\mathfrak{P},\tau)\|,\cdots,\|\mathscr{J}_n(\mathfrak{P},2\mathfrak{J}\tau) - 2\mathfrak{J}\mathscr{J}_n(\mathfrak{P},\tau)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\tau}{2},\tau,\frac{\tau}{2}\right),\cdots,\ \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\tau}{2},\tau,\frac{\tau}{2}\right)\ \Big],\end{aligned} \tag{13}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\tau \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Let $\hbar := (\hbar_1,\cdots,\hbar_n)$, and $\hbar' := (\hbar_1',\cdots,\hbar_n')$.

Next, consider the set

$$\nabla := \{\hbar : (\mathscr{X}' \times \mathscr{X})^n \to \mathscr{X}^n : \hbar(\mathfrak{P},0) = \overbrace{(0,\cdots,0)}^{n}\}$$

and define the generalized metric on ∇

$$\begin{aligned}d(\hbar,\hbar') = \inf\Bigg\{&(\mu_1,\cdots,\mu_n) \in \mathbb{R}^n_{\ge 0} :\\ &\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P},\alpha) - \hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\hbar_n(\mathfrak{P},\alpha) - \hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \mu_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\mu_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],\ \ \forall\alpha \in \mathscr{X},\mathfrak{P} \in \mathscr{X}'\Bigg\},\end{aligned}$$

where as usual, $\inf(\emptyset,\cdots,\emptyset) = (+\infty,\cdots,+\infty)$. It is easy to show that (∇, d) is a complete generalized metric space.

Let $\mathscr{L} := (\mathscr{L}_1,\cdots,\mathscr{L}_n)$. Now we define the linear mapping $\mathscr{L} : \nabla \to \nabla$ s.t.

$$\mathscr{L}_i\hbar_i(\mathfrak{P},\alpha) = 2\mathfrak{J}\hbar_i\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right),\ \ \forall i = 1,\cdots,n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Assume $\hbar, \hbar' \in \nabla$ be given s.t. $d(\hbar,\hbar') = (\varepsilon_1,\cdots,\varepsilon_n)$. Then

$$\begin{aligned}&\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P},\alpha) - \hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\hbar_n(\mathfrak{P},\alpha) - \hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\ &\preceq \operatorname{diag}\Big[\ \varepsilon_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\varepsilon_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Hence

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{L}_1\hbar_1(\mathfrak{P},\alpha)-\mathscr{L}_1\hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\mathscr{L}_n\hbar_n(\mathfrak{P},\alpha)-\mathscr{L}_n\hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\
&=\operatorname{diag}\Big[\ \left\|2\mathfrak{J}\hbar_1\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)-2\mathfrak{J}\hbar_1'\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|,\cdots,\left\|2\mathfrak{J}\hbar_n\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)-2\mathfrak{J}\hbar_n'\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ 2\varepsilon_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right),\cdots,2\varepsilon_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right)\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{4}\varepsilon_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathcal{T}_n}{4}\varepsilon_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. So $d(\hbar,\hbar') = (\varepsilon_1,\cdots,\varepsilon_n)$ implies that $d(\mathscr{L}\hbar(\mathfrak{P},\alpha),\mathscr{L}\hbar'(\mathfrak{P},\alpha)) \preceq (\frac{\mathcal{T}_1}{4}\varepsilon_1,\cdots,\frac{\mathcal{T}_n}{4}\varepsilon_n)$. Hence

$$d(\mathscr{L}\hbar(\mathfrak{P},\alpha),\mathscr{L}\hbar'(\mathfrak{P},\alpha)) \preceq (\frac{\mathcal{T}_1}{4},\cdots,\frac{\mathcal{T}_n}{4})d(\hbar,\hbar')$$

for each $\hbar,\hbar' \in \nabla$. According to (13),

$$\begin{aligned}
&\operatorname{diag}\ \Big[\left\|\mathscr{J}_1(\mathfrak{P},\alpha)-2\mathfrak{J}\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|,\cdots,\left\|\mathscr{J}_n(\mathfrak{P},\alpha)-2\mathfrak{J}\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|\ \Big]\\
&\preceq\operatorname{diag}\ \Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right),\cdots,\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right)\ \Big]\\
&\preceq\operatorname{diag}\ \Big[\frac{\mathcal{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathcal{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$ and so $d(\mathscr{J},\mathscr{L}\mathscr{J}) \preceq (\frac{\mathcal{T}_1}{8},\cdots,\frac{\mathcal{T}_n}{8})$.

By the fixed point alternative we deduce the existence of unique fixed points of $\mathscr{L}_i, (i = 1,\cdots,n)$, that is, the existence of mappings $\mathscr{G}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1,\cdots,n)$, s.t.

$$\mathscr{G}_i(\mathfrak{P},\alpha) = 2\mathfrak{J}\mathscr{G}_i\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right),\ \ \forall i = 1,\cdots,n$$

with the following property: there exist $\mathcal{T}_1,\cdots,\mathcal{T}_n \in (0,\infty)$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\alpha)\|,\cdots,\|\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\alpha)\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \mathcal{T}_1\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\mathcal{T}_n\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Since $\lim_{m\to\infty} d(\mathscr{L}^m\mathscr{J},\mathscr{G}) = (0,\cdots,0)$, in which $\mathscr{J} := (\mathscr{J}_1,\cdots,\mathscr{J}_n)$, $\mathscr{G} := (\mathscr{G}_1,\cdots,\mathscr{G}_n)$, and $\mathscr{L} := (\mathscr{L}_1,\cdots,\mathscr{L}_n)$, then

$$\lim_{m\to\infty} 2^m\mathfrak{J}^m\mathscr{J}_i\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right) = \mathscr{G}_i(\mathfrak{P},\alpha), \quad \forall i = 1,\cdots,n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. In particular,

$$\mathscr{G}_i(\mathfrak{P},\alpha) = \lim_{m\to\infty} 2^m\mathscr{J}_i\left(\mathfrak{P},\frac{\alpha}{2^m}\right) = \mathscr{J}_i(\mathfrak{P},\alpha), \quad \forall i = 1,\cdots,n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, since $\mathscr{J}_i(\mathfrak{P}, 2\alpha) = 2\mathscr{J}_i(\mathfrak{P}, \alpha), (i = 1, \cdots, n)$, for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Also, $d(\mathscr{J}, \mathscr{G}) \preceq (\frac{1}{1-\frac{\mathcal{T}_1}{4}}, \cdots, \frac{1}{1-\frac{\mathcal{T}_n}{4}})d(\mathscr{J}, \mathscr{L}\mathscr{J})$ which implies

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\alpha)\|\ ,\cdots,\ \|\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\alpha)\|\ \Big]\\
&\preceq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2(4-\mathcal{T}_1)}\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathcal{T}_n}{2(4-\mathcal{T}_n)}\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. It follows from (10) and (11) that

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{G}_1(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{G}_1(\mathfrak{P},\alpha+\gamma)-\mathscr{G}_1(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)\|\\
&\quad,\cdots,\ \|\mathscr{G}_n(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{G}_n(\mathfrak{P},\alpha+\gamma)-\mathscr{G}_n(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)\|\ \Big]\\
&=\operatorname{diag}\Big[\ \lim_{m\to\infty}\Big\|2^m\mathfrak{J}^m\Big(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m\mathfrak{J}^m}\right)\\
&\qquad-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)\Big)\Big\|\\
&\quad,\cdots,\ \lim_{m\to\infty}\Big\|2^m\mathfrak{J}^m\Big(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m\mathfrak{J}^m}\right)\\
&\qquad-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)\Big)\Big\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_1\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\quad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_1'\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\quad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\\
&\quad,\cdots,\ \lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_n\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\quad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_n'\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\quad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\ \Big]\\
&=\operatorname{diag}\Big[\ \|\theta_1(\mathscr{G}_1(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)-\mathscr{G}_1(\mathfrak{P},\beta))\|\\
&\qquad+\|\theta_1'(\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)+\mathscr{G}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\gamma))\|\\
&\quad,\cdots,\ \|\theta_n(\mathscr{G}_n(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)-\mathscr{G}_n(\mathfrak{P},\beta))\|\\
&\qquad+\|\theta_n'(\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)+\mathscr{G}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\gamma))\|\ \Big],
\end{aligned}$$

for each $\alpha, \beta, \gamma \in \mathscr{X},\ \mathfrak{P} \in \mathscr{X}'$. According to Lemma 2.1, the mappings $\mathscr{G}_i, (i = 1, \cdots, n)$, are additive.

Putting $\alpha = \gamma = \frac{\tau}{2}$ and $\beta = 0$ in (11), we have

$$\begin{aligned} &\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}\tau) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \tau)\|\ , \cdots ,\ \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}\tau) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \tau)\|\ \Big] \\ &\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1 \le n}\left(\frac{\tau}{2}, 0, \frac{\tau}{2}\right)\ , \cdots ,\ \underbrace{\varphi_{j_n}}_{1\le j_n \le n}\left(\frac{\tau}{2}, 0, \frac{\tau}{2}\right)\ \Big], \end{aligned}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\tau \in \mathscr{X},\ \mathfrak{P} \in \mathscr{X}'$. Thus

$$\begin{aligned} &\operatorname{diag}\Big[\ \|\mathscr{G}_1(\mathfrak{P}, \mathfrak{J}\alpha) - \mathfrak{J}\mathscr{G}_1(\mathfrak{P}, \alpha)\|, \cdots , \|\mathscr{G}_n(\mathfrak{P}, \mathfrak{J}\alpha) - \mathfrak{J}\mathscr{G}_n(\mathfrak{P}, \alpha)\|\ \Big] \\ &= \operatorname{diag}\Big[\ \lim_{m\to\infty} 2^m |\mathfrak{J}|^m \left\| \mathscr{J}_1\left(\mathfrak{P}, \mathfrak{J}\frac{\alpha}{2^m \mathfrak{J}^m}\right) - \mathfrak{J}\mathscr{J}_1\left(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\right)\right\| \\ &\qquad\qquad , \cdots ,\ \lim_{m\to\infty} 2^m |\mathfrak{J}|^m \left\| \mathscr{J}_n\left(\mathfrak{P}, \mathfrak{J}\frac{\alpha}{2^m \mathfrak{J}^m}\right) - \mathfrak{J}\mathscr{J}_n\left(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\right)\right\|\ \Big] \\ &\preceq \operatorname{diag}\Big[\ \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_1}}_{1\le j_1 \le n}\left(\frac{\alpha}{2^{m+1}\mathfrak{J}^m}, 0, \frac{\alpha}{2^{m+1}\mu^m}\right) \\ &\qquad\qquad , \cdots ,\ \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_n}}_{1\le j_n \le n}\left(\frac{\alpha}{2^{m+1}\mathfrak{J}^m}, 0, \frac{\alpha}{2^{m+1}\mu^m}\right)\ \Big] \\ &\preceq \operatorname{diag}\Big[\ \lim_{m\to\infty}\left(\frac{\mathcal{T}_1}{4}\right)^m \underbrace{\varphi_{j_1}}_{1\le j_1 \le n}\left(\frac{\alpha}{2}, 0, \frac{\alpha}{2}\right), \cdots , \lim_{m\to\infty}\left(\frac{\mathcal{T}_n}{4}\right)^m \underbrace{\varphi_{j_n}}_{1\le j_n \le n}\left(\frac{\alpha}{2}, 0, \frac{\alpha}{2}\right)\ \Big], \end{aligned}$$

which tend to zero as $n \to \infty$ and so $\mathscr{G}_i(\mathfrak{P}, \mathfrak{J}\alpha) = \mathfrak{J}\mathscr{G}_i(\mathfrak{P}, \alpha), (i = 1, \ldots, n)$, for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha \in \mathscr{X},\ \mathfrak{P} \in \mathscr{X}'$. Therefore, according to Lemma 2.4, the mappings $\mathscr{G}_i, (i = 1, \ldots, n)$, are $\mathbb{C}$-linear.

Since $\mathscr{J}_i = \mathscr{G}_i, (i = 1, \ldots, n)$, is continuous and $\mathbb{C}$-linear, it follows from (10) and (12) that

$$\begin{aligned}
&\mathrm{diag}\Big[\Big\|[\mathscr{G}_1(\mathfrak{P},\alpha),\mathscr{G}_1(\mathfrak{P},\beta),\mathscr{G}_1(\mathfrak{P},\gamma)] - \mathscr{G}_1(\mathfrak{P},[\mathscr{G}_1(\mathfrak{P},\alpha),\beta,\gamma]) \\
&\quad - \mathscr{G}_1(\mathfrak{P},[\alpha,\mathscr{G}_1(\mathfrak{P},\beta),\gamma]) - \mathscr{G}_1(\mathfrak{P},[\alpha,\beta,\mathscr{G}_1(\mathfrak{P},\gamma)])\Big\|, \cdots, \\
&\Big\|[\mathscr{G}_n(\mathfrak{P},\alpha),\mathscr{G}_n(\mathfrak{P},\beta),\mathscr{G}_n(\mathfrak{P},\gamma)] - \mathscr{G}_n(\mathfrak{P},[\mathscr{G}_n(\mathfrak{P},\alpha),\beta,\gamma]) \\
&\quad - \mathscr{G}_n(\mathfrak{P},[\alpha,\mathscr{G}_n(\mathfrak{P},\beta),\gamma]) - \mathscr{G}_n(\mathfrak{P},[\alpha,\beta,\mathscr{G}_n(\mathfrak{P},\gamma)])\Big\| \Big] \\
&= \mathrm{diag}\Big[\lim_{m\to\infty} \Big\| 2^{3m}\mathfrak{J}^{3m}\Big[\mathscr{J}_1\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_1\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_1\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big] \\
&\quad - 2^m\mathfrak{J}^m\mathscr{G}_1\Big(\mathfrak{P},\Big[\mathscr{J}_1\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\beta,\gamma\Big]\Big) - 2^m\mathfrak{J}^m\mathscr{G}_1\Big(\mathfrak{P},\Big[\alpha,\mathscr{J}_1\Big(\frac{\beta}{2^m\mathfrak{J}^m}\Big),\gamma\Big]\Big) \\
&\quad - 2^m\mathfrak{J}^m\mathscr{G}_1\Big(\mathfrak{P},\Big[\alpha,\beta,\mathscr{J}_1\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big]\Big)\Big\| , \cdots, \\
&\lim_{m\to\infty} \Big\| 2^{3m}\mathfrak{J}^{3m}\Big[\mathscr{J}_n\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_n\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_n\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big] \\
&\quad - 2^m\mathfrak{J}^m\mathscr{G}_n\Big(\mathfrak{P},\Big[\mathscr{J}_n\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\beta,\gamma\Big]\Big) - 2^m\mathfrak{J}^m\mathscr{G}_n\Big(\mathfrak{P},\Big[\alpha,\mathscr{J}_1\Big(\frac{\beta}{2^m\mathfrak{J}^m}\Big),\gamma\Big]\Big) \\
&\quad - 2^m\mathfrak{J}^m\mathscr{G}_n\Big(\mathfrak{P},\Big[\alpha,\beta,\mathscr{J}_n\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big]\Big)\Big\| \Big] \\
&= \mathrm{diag}\Big[\lim_{m\to\infty} 2^{3m}\Big\|\Big[\mathscr{J}_1\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_1\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_1\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big] \\
&\quad - \mathscr{J}_1\Big(\mathfrak{P},\Big[\mathscr{J}_1\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\Big]\Big) - \mathscr{J}_1\Big(\mathfrak{P},\Big[\frac{\alpha}{2^m\mathfrak{J}^m},\mathscr{J}_1\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\frac{\gamma}{2^m\mathfrak{J}^m}\Big]\Big) \\
&\quad - \mathscr{J}_1\Big(\mathfrak{P},\Big[\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\mathcal{F}\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big]\Big)\Big\| , \cdots, \\
&\lim_{m\to\infty} 2^{3m}\Big\|\Big[\mathscr{J}_n\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_n\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\mathscr{J}_n\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big] \\
&\quad - \mathscr{J}_n\Big(\mathfrak{P},\Big[\mathscr{J}_n\Big(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\Big),\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\Big]\Big) - \mathscr{J}_n\Big(\mathfrak{P},\Big[\frac{\alpha}{2^m\mathfrak{J}^m},\mathscr{J}_n\Big(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\Big),\frac{\gamma}{2^m\mathfrak{J}^m}\Big]\Big) \\
&\quad - \mathscr{J}_n\Big(\mathfrak{P},\Big[\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\mathscr{J}_n\Big(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\Big)\Big]\Big)\Big\| \Big] \\
&\preceq \mathrm{diag}\Big[\lim_{m\to\infty} 2^{3m} \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\Big(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\Big),\cdots,\lim_{m\to\infty} 2^{3m} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\Big(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big] \\
&\preceq \mathrm{diag}\Big[\lim_{m\to\infty} \mathcal{T}_1^m \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha,\beta,\gamma),\cdots,\lim_{m\to\infty} \mathcal{T}_n^m \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha,\beta,\gamma) \Big],
\end{aligned}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha,\beta,\gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Since $(\mathcal{T}_1,\cdots,\mathcal{T}_n) < (1,\cdots,1)$, the $\mathbb{C}$-linear mappings $\mathscr{G}_i, (i = 1,\cdots,n)$, are ternary antiderivations. Thus the mappings $\mathscr{J}_i, (i = 1,\cdots,n)$, are ternary antiderivations. □

2.3 Super-multi-stability of continuous ternary antiderivations in ternary Banach algebras

In this subsection, we investigate the super-multi-stability of continuous ternary antiderivations in ternary Banach algebras.

Theorem 2.6. *Let* $\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n},\cdots,\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}:\mathscr{X}^3\to[0,\infty)$ *be functions. If there exist* $(\mathcal{T}_1,\cdots,\mathcal{T}_n)<(1,\cdots,1)$ *with satisfying*

$$\begin{aligned} &diag\Big[\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\alpha}{\mathfrak{J}},\frac{\beta}{\mathfrak{J}},\frac{\gamma}{\mathfrak{J}}\right),\cdots,\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\alpha}{\mathfrak{J}},\frac{\beta}{\mathfrak{J}},\frac{\gamma}{\mathfrak{J}}\right)\Big] \\ &\preceq diag\Big[\frac{\mathcal{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(2\alpha,2\beta,2\gamma),\cdots,\frac{\mathcal{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(2\alpha,2\beta,2\gamma)\Big], \end{aligned} \tag{14}$$

for each $\mathfrak{J}$ *with* $|\mathfrak{J}|<1$ *(resp.* $|\mathfrak{J}|>1$*) and each* $\alpha,\beta,\gamma\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$*. Let* $\mathscr{J}_i:\mathscr{X}'\times\mathscr{X}\to\mathscr{X},(i=1,\cdots,n)$*, be mappings satisfying*

$$\begin{aligned} &diag\Big[\|\mathscr{J}_1(\mathfrak{P},\mathfrak{J}(\alpha+\beta+\gamma))-\mathfrak{J}\mathscr{J}_1(\mathfrak{P},\alpha+\gamma)-\mathfrak{J}\mathscr{J}_1(\mathfrak{P},\beta-\alpha+\gamma)-\mathfrak{J}\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)\| \\ &\quad,\cdots,\ \|\mathscr{J}_n(\mathfrak{P},\mathfrak{J}(\alpha+\beta+\gamma))-\mathfrak{J}\mathscr{J}_n(\mathfrak{P},\alpha+\gamma)-\mathfrak{J}\mathscr{J}_n(\mathfrak{P},\beta-\alpha+\gamma)-\mathfrak{J}\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)\|\Big] \\ &\preceq diag\Big[\|\theta_1(\mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_1(\mathfrak{P},\beta))\| \\ &\quad+\|\theta_1'(\mathscr{J}_1(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{J}_1(\mathfrak{P},\gamma))\|+\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(\alpha,\beta,\gamma) \\ &\quad,\cdots,\ \|\theta_n(\mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_n(\mathfrak{P},\beta))\| \\ &\quad+\|\theta_n'(\mathscr{J}_n(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{J}_1(\mathfrak{P},\gamma))\|+\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(\alpha,\beta,\gamma)\Big], \end{aligned} \tag{15}$$

and (12)*. If* $\mathscr{J}_i,(i=1,\cdots,n)$*, are continuous and* $\mathscr{J}_i(\mathfrak{P},2\alpha)=2\mathscr{J}_i(\mathfrak{P},\alpha)$ *for each* $\alpha\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$*, then the mappings* $\mathscr{J}_i:\mathscr{X}'\times\mathscr{X}\to\mathscr{X},(i=1,\cdots,n)$*, are ternary antiderivations.*

Proof. Suppose $\mathfrak{J}\in\mathbb{T}^1$. Then there exists a sequence $\{\mathfrak{J}_m\}_{m=1}^{\infty}$ with $|\mathfrak{J}_m|<1$ (resp. $|\mathfrak{J}_m|>1$) s.t.

$$\lim_{m\to\infty}\mathfrak{J}_m=\mathfrak{J}.$$

By (14) and (15) we get

$$\begin{aligned} &\mathrm{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}\left(\frac{\alpha}{\mathfrak{J}_m},\frac{\beta}{\mathfrak{J}_m},\frac{\gamma}{\mathfrak{J}_m}\right),\cdots,\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}\left(\frac{\alpha}{\mathfrak{J}_m},\frac{\beta}{\mathfrak{J}_m},\frac{\gamma}{\mathfrak{J}_m}\right)\Big] \\ &\preceq \mathrm{diag}\Big[\frac{\mathcal{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\leq j_1\leq n}(2\alpha,2\beta,2\gamma),\cdots,\frac{\mathcal{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\leq j_n\leq n}(2\alpha,2\beta,2\gamma)\Big] \end{aligned}$$

for each $\mathfrak{I}_m$ with $|\mathfrak{I}_m| < 1$ (resp. $|\mathfrak{I}_m| > 1$) and each $\alpha, \beta, \gamma \in \mathscr{X}$, and

$$\begin{aligned}
&\mathrm{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \mathfrak{I}_m(\alpha+\beta+\gamma)) - \mathfrak{I}_m \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}_m \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}_m \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\| \\
&\quad , \cdots , \ \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{I}_m(\alpha+\beta+\gamma)) - \mathfrak{I}_m \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}_m \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}_m \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \ \Big] \\
&\preceq \mathrm{diag}\Big[\ \|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha, \beta, \gamma) \\
&\qquad , \cdots , \ \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha, \beta, \gamma) \ \Big],
\end{aligned}$$

for each positive integers m and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Passing to the limit as $n \to \infty$, and using the continuity of $\underbrace{\varphi_{j_1}}_{1\le j_1\le n}, \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}$, $\mathscr{J}_1, \cdots, \mathscr{J}_n$ and $\|\cdot\|$, we obtain

$$\begin{aligned}
&\mathrm{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}\left(\frac{\alpha}{\mathfrak{I}}, \frac{\beta}{\mathfrak{I}}, \frac{\gamma}{\mathfrak{I}}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}\left(\frac{\alpha}{\mathfrak{I}}, \frac{\beta}{\mathfrak{I}}, \frac{\gamma}{\mathfrak{I}}\right) \ \Big] \\
&\preceq \mathrm{diag}\Big[\ \frac{\mathcal{T}_1}{8} \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathcal{T}_n}{8} \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(2\alpha, 2\beta, 2\gamma) \ \Big],
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \mathfrak{I}(\alpha+\beta+\gamma)) - \mathfrak{I} \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I} \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I} \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\| \\
&\quad , \cdots , \ \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{I}(\alpha+\beta+\gamma)) - \mathfrak{I} \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I} \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I} \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \ \Big] \\
&\preceq \mathrm{diag}\Big[\ \|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha, \beta, \gamma) \\
&\qquad , \cdots , \ \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha, \beta, \gamma) \ \Big]
\end{aligned}$$

for each $\mathfrak{I} \in \mathbb{T}^1$ and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Thus, by the same reasoning as in the proof of Theorem 2.5, the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are ternary antiderivations. □

2.4 Application

Here, let $n = 7$.

Corollary 2.7. *Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying*

$$\begin{aligned}
&diag\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots,\\
&\qquad \|\mathscr{J}_n(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\|\ \Big]\\
&\preceq diag\Big[\ \|\theta_1(\mathscr{J}_1(\alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\|\\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}^{\circledS}}_{1\le j_1\le n}(\|[\alpha,\beta,\gamma]\|), \cdots,\\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\|\\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}^{\circledS}}_{1\le j_n\le n}(\|[\alpha,\beta,\gamma]\|)\ \Big],
\end{aligned}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Then there exists unique additive mappings $\mathscr{J}_i' : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, s.t.

$$\begin{aligned}
&diag\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1'(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n'(\mathfrak{P}, \alpha)\|\ \Big]\\
&\preceq diag\Big[\ \underbrace{\varphi_{j_1}^{\circledS}}_{1\le j_1\le n}(\|[\alpha,\alpha,\alpha]\|), \cdots, \underbrace{\varphi_{j_n}^{\circledS}}_{1\le j_n\le n}(\|[\alpha,\alpha,\alpha]\|)\ \Big], \qquad \forall i = 1, \cdots, n
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Proof. The proof follows from Theorem 2.2 by letting $\mathcal{T}_i = \frac{2}{3}, (i = 1, \cdots, n)$, and

$$\text{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha,\beta,\gamma), \cdots, \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha,\beta,\gamma)\ \Big] := \text{diag}\Big[\ \underbrace{\varphi_{j_1}^{\circledS}}_{1\le j_1\le n}(\|[\alpha,\beta,\gamma]\|), \cdots, \underbrace{\varphi_{j_n}^{\circledS}}_{1\le j_n\le n}(\|[\alpha,\beta,\gamma]\|)\ \Big],$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$. □

Corollary 2.8. *Assume $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, be mappings satisfying*

$$\begin{aligned}
&diag\Big[\ \|\mathscr{J}_1(\mathfrak{I}(\mathfrak{P}, \alpha+\beta+\gamma)) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots,\\
&\qquad \|\mathscr{J}_n(\mathfrak{I}(\mathfrak{P}, \alpha+\beta+\gamma)) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\|\ \Big]\\
&\preceq diag\Big[\ \|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\|\\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}^{\circledS}}_{1\le j_1\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big), \cdots,\\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\|\\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}^{\circledS}}_{1\le j_n\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ \Big],
\end{aligned}$$

and

$$\begin{aligned} diag\Big[& \Big\| [\mathscr{J}_1(\mathfrak{P},\alpha), \mathscr{J}_1(\mathfrak{P},\beta), \mathscr{J}_1(\mathfrak{P},\gamma)] - \mathscr{J}_1\Big(\mathfrak{P}, [\mathscr{J}_1(\mathfrak{P},\alpha),\beta,\gamma]\Big) \\ & + \mathscr{J}_1\Big(\mathfrak{P}, [\alpha, \mathscr{J}_1(\mathfrak{P},\beta),\gamma]\Big) + \mathscr{J}_1\Big(\mathfrak{P}, [\alpha,\beta, \mathscr{J}_1(\mathfrak{P},\gamma)]\Big)\Big\| \\ & ,\cdots, \quad \Big\| [\mathscr{J}_n(\mathfrak{P},\alpha), \mathscr{J}_n(\mathfrak{P},\beta), \mathscr{J}_n(\mathfrak{P},\gamma)] - \mathscr{J}_n\Big(\mathfrak{P}, [\mathscr{J}_n(\mathfrak{P},\alpha),\beta,\gamma]\Big) \\ & + \mathscr{J}_n\Big(\mathfrak{P}, [\alpha, \mathscr{J}_n(\mathfrak{P},\beta),\gamma]\Big) + \mathscr{J}_n\Big(\mathfrak{P}, [\alpha,\beta, \mathscr{J}_n(\mathfrak{P},\gamma)]\Big)\Big\| \ \Big] \\ \preceq diag\Big[& \underbrace{\varphi^{\circledS}_{j_1}}_{1\le j_1\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big) \ ,\cdots, \ \underbrace{\varphi^{\circledS}_{j_n}}_{1\le j_n\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ \Big], \end{aligned}$$

for each $\alpha,\beta,\gamma\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$. *If* $\mathscr{J}_i,(i=1,\cdots,n)$, *are continuous and* $\mathscr{J}_i(\mathfrak{P},2\alpha)=2\mathscr{J}_i(\mathfrak{P},\alpha),(i=1,\cdots,n)$, *for each* $\alpha\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$, *then the mappings* $\mathscr{J}_i:\mathscr{X}'\times\mathscr{X}\to\mathscr{X},(i=1,\cdots,n)$, *are ternary antiderivations.*

Proof. The proof follows from Theorem 2.5 by letting $(\mathcal{T}_1,\cdots,\mathcal{T}_n)=(\frac{32}{33},\cdots,\frac{32}{33})$ and

$$\begin{aligned} &\mathrm{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\le j_1\le n}(\alpha,\beta,\gamma),\ldots,\ \underbrace{\varphi_{j_n}}_{1\le j_n\le n}(\alpha,\beta,\gamma)\ \Big] \\ &=: \mathrm{diag}\Big[\ \underbrace{\varphi^{\circledS}_{j_1}}_{1\le j_1\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big),\ldots,\ \underbrace{\varphi^{\circledS}_{j_n}}_{1\le j_n\le n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ \Big] \end{aligned}$$

for each $\alpha,\beta,\gamma\in\mathscr{X},\mathfrak{P}\in\mathscr{X}'$. □

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