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Some remarks on regular p-table algebra

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Abstract

Let (A, B) be a standard p-table algebra and assume that $O^{\vartheta}(B) \subseteq O_{\vartheta}(B)$. Let b be an element in B. Then we have $b\bar{b}b = b$. Since regular relations satisfy $b\bar{b}b = b$, we have investigated regular relations of standard p-table algebra. Also, we give the structure constants of standard regular p-table algebra.

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1 Introduction

A table algebra is a finite dimensional over the complex numbers with a distinguished basis.The information in this section regarding table algebras can be found in [3] and the information regarding association schemes can be found in [6].

Definition 1.1. Let B be a basis of a finite dimensional associative and commutative algebra A over the complex field $\mathbb{C}$ with the identity 1. Then the pair (A, B) is called a table algebra if $1 \in B$ and the following conditions hold:

- For all $a, b \in B, ab = \sum_{c \in B} \lambda_{abc} c$, with λ_{abc} a nonnegative real number.
- There is an algebra automorphism $^{-}$ of A whose order divides 2, such that $\bar{B} = B$.
- $\lambda_{ab1} \neq 0 \leftrightarrow b = \bar{a}$.

Let (A, B) be a table algebra. There exist unique degree map $|\ \ | : A \to C$ such that $|b_i| = |\bar{b}_i|$ for all $b_i \in B$ (see [1]). If for any $b \in B, |b| = \lambda_{b\bar{b}1}$, then (A, B) is called a standard table algebra. If all structure constants and degrees $|b|$ for $b \in B$ are nonnegative integers, then (A, B) is called an integral table algebra.

For any table algebra (A, B), there is a positive definite Hermitian form $(,)$ on A(that is, the form $(,)$ is biadditive, and for all $a, b \in A$ and $\gamma \in \mathbb{C}, (\gamma a, b) = \gamma(a, b),\ (a, b) = \overline{(b, a)}$ and $(a, a) > 0$ if $a \neq 0$) with the following properties [3, Proposition 2.1]:

- $(ab, c) = (b, \bar{a}c) = (a, c\bar{b})$

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- $(b, b) = \lambda_{b\bar{b}1}$.

Example 1.2. The Bose-Mesner algebra of an association scheme is a standard table algebra. Bose-Mesner algebra has a basis A_i consisting of $0, 1$-matrices such that $A_0 = I$ and $A_iA_j = \sum \lambda_{ijk}A_k$. The set of these adjacency matrices forms a basis for the algebra that it generates, and Definition 1.1 is satisfied, where the automorphism is matrix transpose. The degree of each adjacency matrix is the sum over any of its rows, and this well defined positive integer is the valency of the corresponding relation.

Let (A, B) be a table algebra. For each $b \in B$, the stable degree of b is defined as

$$\sigma(b) := \frac{|b|^2}{\lambda_{b\bar{b}1}}.$$

For any subset $\mathbf{S} \subseteq B$, by the order of S we mean

$$o(S) := \sum_{b \in \mathrm{S}} \sigma(b).$$

Definition 1.3. Let p be a prime number. Let (A, B) be a table algebra and let $S \subseteq B$. Then S is called p-valenced if for all $b \in S, \delta(b)$ is a power of p. Also, S is called a p-subset of B if S is p-valenced and $o(S)$ is a power of p. If (A, B) is standard and B itself is a p-subset, then (A, B) is called a p-standard table algebra.

A nonempty subset T of B is called a closed subset of B if $\overline{T}T \subseteq T$. For any $b \in B$, the closed subset generated by $\{b\}$ is denoted by $< b >$. The thin residues and the thin radicals of closed subsets of association schemes are studied in [6]. These concepts can be defined similarly for closed subsets of table algebras. Let N be a closed subset of B. Then the thin residue of N is defined by

$$O^{\vartheta}(N) = \left\langle \bigcup_{b \in N} \mathrm{Supp}_B < b\bar{b} > \right\rangle.$$

Define $O_{\vartheta}(B) =$, the thin radical of B, is the set of all thin elements of B.

2 Main results

The following lemma has already proved for association scheme in [7]. Now we prove it for table algebras similarly.

Lemma 2.1. *Let (A, B) be a standard table algebra which is $O^{\vartheta}(B) \subseteq O_{\vartheta}(B)$. Let a and b be elements in B and $T_a = \{b \in O_{\vartheta}(B) \mid ab = b\}$. Let $c \in \mathrm{Supp}_B(ab)$. Then*

(i) We have $\mathrm{Supp}_B(ab) = \mathrm{Supp}_B(cT_b)$.

(ii) We have $|T_b| = |ab|\,|T_b \cap T_c|$.

(iii) For each element d in ab, we have $T_c = T_d$.

Proof. (i) Let $d \in \mathrm{Supp}_B(ab)$. Then $a \in \mathrm{Supp}_B(d\bar{b})$. By assumption $c \in \mathrm{Supp}_B(ab)$. Thus

$$c \in \mathrm{Supp}_B(d\bar{b}b) = \mathrm{Supp}_B(dT_b)$$

and so $d \in \mathrm{Supp}_B\,(cT_b)$. Also, we have

$$\mathrm{Supp}_B\,(cT_b) \subseteq \mathrm{Supp}_B\,(abT_b) \subseteq \mathrm{Supp}_B(ab).$$

(ii) T_b acts transitively on $\mathrm{Supp}_B(ab)$ by right multiplication. Thus

$$\frac{|T_b|}{|T_b \cap T_c|} = |ab|.$$

(iii) Let $d \in \mathrm{Supp}_B(ab)$. Then, by (ii), $d \in \mathrm{Supp}_B\,(cT_b)$. So, there exists an element $t \in T_b$ such that $ct = d$. Thus

$$T_d = \bar{d}d = \bar{t}\bar{c}ct = \bar{t}T_c t$$

and

$$t \in T_b \subseteq O_\vartheta(d) \subseteq K_d\,(T_c)\,.$$

Thus, $T_d = T_c$.

□

Theorem 2.2. *Let (A,B) be a standard integral p-table algebra with $O^\vartheta(B) \subseteq O_\vartheta(B)$ and $|O^\vartheta(B)| = p$ and $|O_\vartheta(B)| = p^2$ for some prime number p. If (A,B) is regular, then we have the following structure constants:*

- $b\bar{b} = pb_0 + \ldots + pb_p, \quad b_i \in O^\vartheta(B), 1 \le i \le p.$
- $b_i b_j = pb_k.$
- $b_i^2 = p\bar{b}_i$ *or* $b_i^2 = pb_0 + \ldots + pb_p.$

Example 2.3. Note that if $\mathcal{C}$ is a 2-scheme of degree 8 whose degrees of thin radical and thin residue are 4 and 2 respectively. A regular association 2-scheme of order 8, No. 13 in Hanaki's classification of association schemes [5] such that

$$\sum_{i=0}^{5} iA_i = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 5 \\ 1 & 0 & 3 & 2 & 4 & 4 & 5 & 5 \\ 2 & 3 & 0 & 1 & 5 & 5 & 4 & 4 \\ 3 & 2 & 1 & 0 & 5 & 5 & 4 & 4 \\ 4 & 4 & 5 & 5 & 0 & 1 & 2 & 3 \\ 4 & 4 & 5 & 5 & 1 & 0 & 3 & 2 \\ 5 & 5 & 4 & 4 & 2 & 3 & 0 & 1 \\ 5 & 5 & 4 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

where A_i is the adjacency matrix of basis relations of $\mathcal{C}$. Let (A,B) be the Bose-Mesner algebra of the association scheme $\mathcal{C}$. Then (A,B) is a noncommutative table algebra. It is easy to check that (A,B) has the following structure constants:

$$A_1^2 = A_2^2 = A_3^2 = A_0, A_4^2 = A_5^2 = 2A_0 + 2A_1, A_4A_5 = 2A_2 + 2A_3, A_1A_2 = A_3,$$
$$A_1A_3 = A_2, A_1A_4 = A_4, A_1A_5 = A_5, A_2A_4 = A_3A_4 = A_5$$

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