A combinatorial calculation on the derangement problem

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ABSTRACT

Assume that $\sigma \epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\dots,n\}$. Consider that A is a subset of $\{1,2,3,\dots,n\}$. We say that σ is a derangement on A, if for any $\boldsymbol{i \epsilon A}$**,** we have $\sigma(i) \neq \boldsymbol{i}$**.** A derangement is an arrangement of objects such that none of the objects are in their rightful place. A derangement can also be called a permutation with no fixed points. If we choose a random permutation, the probability that it is a derangement is close to 1/e.

In this article, we intend to achieve new relations in this field by using relations and coefficients of binomial expansion.

Assume that $\mathrm{D_n}$ is the number of derangement on n elements then:

$$1 - \lim_{n\to\infty} \frac{D_n}{n!} = \sum_{k=0}^{\infty} \frac{1}{(2k+2)[(2k)!]}$$

In fact, these calculations lead us to find a generating function.

KEYWORDS: The Derangement problem, Combinatorial calculation, Euler's number, Binomial coefficient.

1 INTRODUCTION

A derangement is an arrangement of objects such that none of the objects are in their rightful place. A derangement can also be called a permutation with no fixed points. If we choose a random permutation, the probability that it is a derangement is close to 1/e. Another version of the problem arises when we ask for the number of ways n letters, each addressed to a different person, can be placed in n pre-addressed envelopes so that no letter appears in the correctly addressed envelope.

The derangement problem was formulated by P. R. de Montmort in 1708, and solved by him in 1713 (de Montmort 1713-1714). The number of derangements of an n-element set is called the n -th derangement number or rencontres number, or the subfactorial of n and is sometimes denoted D_n. This number satisfies the recurrences

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Also, it is well-known that

$$\lim_{n\to\infty}\frac{D_n}{n!} = e^{-1} = 0.3678\,...$$

When D_n is the number of derangement on n elements then:

$$|D_n| = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

The interesting this is that the number e itself also has applications in probability theory, in a way that is not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Gordon and McMahon noted that the number of derangements in the hyperoctahedral group gives the rising 2-binomial transform of the derangement numbers for S_n. More generally, they shows that the cyclic derangement numbers give a mixed version of the rising r-binomial transform and falling $(r - 1)$ binomial transform of D_n. This new hybrid k-binomial transform may share many of the nice properties of Spivey and Steil's transforms, including Hankel invariance and/or a simple description of the change in the exponential generating function. Further, it could be interesting to evaluate the expression for negative or even non-integer values of k. For instance, taking $k = 1/2$ gives the binomial mean transform which is of some interest.

In this manner, we obtained the following information in previous works.

Theorem A. Assume that $\sigma \epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\dots,n\}$. Consider that k is an integer such that $1 \le k < n$. We define

$$X_k = \{\sigma \epsilon S_n \mid \sigma(i+1) + k \neq \sigma(i)\ for\ 1 \le i \le n-1\}.$$

Also, assume that $s_k = |X_k|$, the cardinal number of X_k.

Then,

$$s_k = \binom{k}{1} s_{k-1} + \cdots + \binom{k}{k-1} s_1 + \binom{k}{k} s_0.$$

When, $D_n = s_0$.

Theorem B. For all $n > 2$ and $r > 0$, we have

$$D_r(n) = rD_{r-1}(n-1) + (n-1)D_r(n-2) + (n+r-1)D_r(n-1)$$

Also, we define a new case of derangement and obtain some result about this special case of derangement.

THEOREM C. Assume that A is a subset of $\{1,2,3,\dots,n\}$ and consider $\sigma\boldsymbol{\epsilon}\boldsymbol{S_n}$ is a derangement on A. Also assume that $|A| = m \le n$ and D_m is the set of all derangement on A, then:

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n-k}{m-k}(m-k)!$$

For more result, see [1], [2], [3], [4], [5] and [6].

2 MAIN RESULT

As mentioned in the introduction section:

$$|D_n| = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

In this section, we intend to achieve new relations in this field by using relations and coefficients of binomial expansion.

Assume that $\sigma\epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\dots,n\}$. Consider that A is a subset of $\{1,2,3,\dots,n\}$. We say that σ is a derangement on A, if for any $\boldsymbol{i\epsilon A},$ we have $\sigma(i) \neq \boldsymbol{i}.$ Now, by this definition we have the following result.

MAIN THEOREM. Assume that $\mathrm{D_n}$ is the number of derangement on n elements then:

$$1-\lim_{n\to\infty}\frac{D_n}{n!} = \sum_{k=0}^{\infty}\frac{1}{(2k+2)[(2k)!]}$$

In fact, these calculations lead us to find a generating function.

Proof. Notice that

$$\sum_{k=0}^{\infty}\frac{2k+1}{(2k+2)!} = \sum_{k=0}^{\infty}\frac{1}{(2k+2)[(2k)!]}$$

$$= \sum_{k=1}^{\infty}\frac{2k-1}{(2k)!}$$

$$= \sum_{k=1}^{\infty}\frac{1}{(2k-1)!} - \sum_{k=1}^{\infty}\frac{1}{(2k)!}$$

$$= \left(\sum_{k:odd}^{\infty}\frac{1}{k!} - \sum_{k:even}^{\infty}\frac{1}{k!}\right) + 1$$

$$= 1 - \left(\sum_{k:even}^{\infty} \frac{1}{k!} - \sum_{k:odd}^{\infty} \frac{1}{k!} \right)$$
$$= 1 - \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right) = \frac{e-1}{e}$$
$$= 1 - \lim_{n\to\infty} \frac{D_n}{n!}$$

In fact, these calculations lead us to find a generating function. We have:

$$\sum_{k=0}^{\infty} \frac{2k+1}{(2k+2)!} = \sum_{k=0}^{\infty} \frac{1}{(2k+2)[(2k)!]}$$
$$= 1 - \left(\sum_{k:even}^{\infty} \frac{1}{k!} - \sum_{k:odd}^{\infty} \frac{1}{k!} \right)$$

$$= 1 - \left(\sum_{k:even}^{\infty} \frac{(x)^k}{k!} - \sum_{k:odd}^{\infty} \frac{(x)^k}{k!} \right)_{x=1}$$
$$= 1 - \left(\sum_{k=0}^{\infty} \frac{(x)}{(2k)!}^{2k} - x \sum_{k=0}^{\infty} \frac{(x)}{2k+1!}^{2k+1} \right)_{x=1}$$
$$= 1 - (coshx - xsinhx)_{x=1}$$

As we claimed.

□

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