



Some Conjectures on average of Fibonacci and Lucas sequences

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Abstract

The arithmetic mean of the first n Fibonacci numbers is not an integer for all n . However, for some values of n we can observe that it is an integer. In this paper we consider the sequence of integers n for that the average of the first n Fibonacci numbers is an integer. We prove some interesting properties and present two related conjectures.

Keywords: Fibonacci number, Lucas number, Pisano period, rank of appearance, restricted period.)

AMS Mathematical Subject Classification [2010]: 13D45, 39B42

1 Introduction

The Fibonacci sequence $(F_n)_{n \geq 0}$ is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, A000045 [11]. Fibonacci numbers have been extensively studied [5, 6]. Numerous fascinating properties are known as, for instance, their close relation to the binomial coefficient:

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

The average of the first n terms of the Fibonacci sequence is not always an integer. For instance, for $n = 3$ we have $\frac{1+1+2}{3} = \frac{4}{3}$, but for $n = 1, 2, 24, 48, \dots$ are $\left(\frac{1}{n} \sum_{i=1}^n F_i\right)_{n \geq 1}$ integers.

In this paper, we explore the following question: Which terms of the sequence

$$A_F(n) = \left(\frac{1}{n} \sum_{i=1}^n F_i\right)_{n \geq 1} \tag{1}$$

are integers?

We give a characterization for the values of n for that $A_F(n)$ is an integer. Moreover, we present a construction of finding infinitely many n that satisfy the given conditions. Further, we show that there are infinitely many n for that 6 is a divisor of the sum of the first n Fibonacci numbers.

Finally, we show that $A_F(p)$ is not an integer if p is an odd prime number.

Our work is based on the results in [6, 7, 11, 10, 13].

¹speaker

2 Main results

First, we recall some definitions and important theorems [1, 4, 12]. A fundamental identity that we use in this paper is [6, Theorem 5.1]

$$\sum_{i=1}^n F_i = F_{n+2} - 1. \quad (2)$$

The Lucas numbers $(L_n)_{n \geq 0}$, are defined by the same recurrence relation as the Fibonacci numbers with different initial values (see A000032).

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n \geq 2.$$

The following relations between Fibonacci numbers and Lucas numbers can be found in [6]:

$$F_{4k+1} - 1 = F_{2k}L_{2k+1}, \quad (3)$$

$$F_{4k+2} - 1 = F_{2k}L_{2k+2}, \quad (4)$$

$$F_{4k+3} - 1 = F_{2k+2}L_{2k+1}, \quad (5)$$

$$F_{2k} = F_kL_{2k}. \quad (6)$$

An integer a is called a *quadratic residue modulo p* (with $p > 2$) if $p \nmid a$ and there exists an integer b such that $a \equiv b^2 \pmod{p}$. Otherwise, it is called a *non-quadratic residue modulo p* .

Let p be an odd prime number. The *Legendre symbol* is a function of a and p defined as

$$\left(\frac{a}{p}\right) = \begin{cases} +1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a non-quadratic residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

We note that for a prime number p the Legendre symbol, $\left(\frac{5}{p}\right)$, is equal to

$$\left(\frac{5}{p}\right) = \begin{cases} +1, & \text{if } p \equiv \pm 1 \pmod{5}, \\ 0, & \text{if } p \equiv 0 \pmod{5}, \\ -1, & \text{if } p \equiv \pm 2 \pmod{5}. \end{cases}$$

Consider the sequence of the Fibonacci numbers modulo 8:

$$0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, \dots$$

We observe that the reduced sequence is periodic.

Lagrange [7] showed that this property is true in general, i.e., that the Fibonacci sequence is periodic modulo m for any positive integers $m > 1$.

Definition 2.1. For a given positive integer m , we call the least integer such that $(F_n, F_{n+1}) \equiv (0, 1) \pmod{m}$ the *(Pisano) period of the Fibonacci sequence modulo m* and denote it by $\pi(m)$.

The first few values of $\pi(n)$ are given as the sequence A001175 in [11].

We recall as a lemma the fixed point theorem of Fulton and Morris [4].

Lemma 2.2 (Fixed Point Theorem [4]). *Let m be a positive integer greater than 1. Then $\pi(m) = m$ if and only if $m = (24)5^{\lambda-1}$ for some $\lambda > 0$.*

For instance, with $m = 8$ we have $\pi(8) = 12$ and $\alpha(8) = 6$. The 12 terms in the period form two sets of 6 terms. The terms of the second half are 5 times the corresponding terms in the first half (mod 8). For the Lucas sequence $F_n = U_n(P, Q)$; Robinson [10], we have $t \equiv F_{\alpha(m)-1}(-Q) \pmod{m}$ is the multiplier between consecutive parts of length $\alpha(m)$ of the period. If the (mod m) order of t is r then $\pi(m) = r\alpha(m)$. Here $F_n = U_n(1, -1)$, $(P, Q) = (1, -1)$, $\alpha(8) = 6$, $t = 5$, $r = 2$; thus $\pi(8) = 2 \cdot 6 = 12$; Robinson [10]. The 12 terms in the period form two sets of 6 terms. The terms of the second half are 5 times the corresponding terms in the first half (modulo 8). The next definition is

Definition 2.3. For a given positive integer, we call the least integer such that $(F_n, F_{n+1}) \equiv \sigma(0, 1) \pmod{m}$ for some positive integer σ the *restricted period of the Fibonacci sequence modulo m* and denote it by $\alpha(m)$.

Robinson [10] showed the following theorems.

Theorem 2.4. *i) $m \mid F_n$ if and only if $\alpha(m) \mid n$, and*

ii) $m \mid F_n$ and $m \mid F_{n+1} - 1$ if and only if $\pi(m) \mid n$.

Theorem 2.5. *If p is a prime, then*

i) $\alpha(p) \mid (p - \left(\frac{5}{p}\right))$,

ii) if $p \equiv \pm 1 \pmod{5}$, then $\pi(p) \mid (p - 1)$, and

iii) if $p \equiv \pm 2 \pmod{5}$, then $\pi(p) \mid 2(p + 1)$.

The exponent of the multiplier of the Fibonacci sequence modulo p , $t \equiv F_{\alpha(p)-1} \pmod{p}$ is $\frac{\pi(p)}{\alpha(p)}$ and can only take the values 1, 2 and 4.

For a positive integer n and a prime p , the p -adic valuation of n , $\nu_p(n)$, is the exponent of the highest power of p that divides n .

Legendre's classical formula for the p -adic valuation of the factorials is well known:

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

We recall *Lengyel's lemma* [8] about the p -adic evaluation of Fibonacci numbers in cases $p = 2, 3$ and 5.

Lemma 2.6 ([8], Lemmas 1 and 2). *For all $n \geq 0$, we have $\nu_5(F_n) = \nu_5(n)$. On the other hand,*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ 1, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

and

$$\nu_3(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{4}; \\ \nu_3(n) + 1, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

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