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A note on rainbow dominating functions in claw-free graphs

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Abstract

Let $G=(V(G),E(G))$ be a graph. A function $f:V(G)\rightarrow \mathbb{P}(\{1,2\})$ is a 2-rainbow dominating function if for every vertex v with $f(v)=\emptyset$, $f\big(N(v)\big)=\{1,2\}$. An outer-independent 2-rainbow dominating function (OI2RD function) of G is a 2-rainbow dominating function f for which the set of all $v\in V(G)$ with $f(v)=\emptyset$ is independent. The outer independent 2-rainbow domination number (OI2RD number) $\gamma_{oir2}(G)$ is the minimum weight of an OI2RD function of G.

In this paper, we first prove that $n/2$ is a lower bound on the OI2RD number of a connected claw-free graph of order n and characterize all such graphs for which the equality holds, solving an open problem given in an earlier paper.

Keywords: Outer-independent rainbow domination, claw-free graphs, k-unit

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1 Introduction

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [5] as a reference for terminology and notation which are not defined here. A subset $S\subseteq V(G)$ is said to be a *dominating set* in G if each vertex not in S is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Domination presents a model for a situation in which every empty location needs to be protected by a guard occupying a neighboring location. A generalization of domination was proposed in [1], where different types of guards are deployed, and the empty locations must have all types of guards in their neighborhoods. This led to the definition of k-rainbow domination. Indeed, a function $f:V(G)\rightarrow \mathbb{P}(\{1,\cdots,k\})$ is a *k-rainbow dominating function* (kRD function) if for every vertex v with $f(v)=\emptyset$, $f(N(v))=\{1,\cdots,k\}$. The *k-rainbow domination number* $\gamma_{rk}(G)$ is the minimum weight of $\sum_{v\in V(G)}|f(v)|$ taken over all kRD functions of G.

The existence of two adjacent locations with no guards can jeopardize them. Indeed, they would be considered more vulnerable. One improved situation for a location with no guards is to be surrounded by locations in which guards are stationed. This motivates us to consider a kRD function f for which the set of vertices assigned $\emptyset$ under f is independent. In fact, a function f is an *outer independent k-rainbow dominating function* (OIkRD function) of G if f is a kRD function and the set of vertices with weight

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$\emptyset$ is independent. The *outer independent k-rainbow domination number* (OIkRD number) $\gamma_{oirk}(G)$ is the minimum weight of an OIkRD function of G. An OIkRD function of weight $\gamma_{oirk}(G)$ is called a $\gamma_{oirk}(G)$-function. This concept was first introduced by Kang et al. [3] and studied in [2, 4]. Mansouri and Mojdeh [4] showed that the problem of computing the OI2RD number is NP-hard even when restricted to planar graphs with maximum degree at most four and triangle-free graphs.

In this paper, emphasizing the case $k = 2$, we first provide a characterization of all connected claw-free graphs whose OI2RD numbers are equal to half of their orders, solving an open problem from [4].

For any function $f : V(G) \to \mathbb{P}(\{1,2\})$, we let $V_\emptyset$, $V_{\{1\}}$, $V_{\{2\}}$ and $V_{\{1,2\}}$ stand for the set of vertices assigned with $\emptyset$, $\{1\}$, $\{2\}$ and $\{1,2\}$ under f, respectively, and write $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$. Note that $w(f) = |V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|$ is the weight of f.

2 Main result

Mansouri and Mojdeh [4] proved that the OI2RD number of a $K_{1,r}$-free graph G of order n with s' strong support vertices can be bounded from below by $2(n + s')/(1 + r)$. They also posed the open problem of characterizing all $K_{1,r}$-free (or at least claw-free) graphs for which the lower bound holds with equality. Our aim in this section is to solve the problem for the claw-free graphs (that is, the case $r = 3$). By the definition, it only suffices to solve the problem when G is connected. Note that $s' = 0$, unless $G = P_3$ for some $1 \le i \le t$. But $\gamma_{oir2}(P_3) = (|V(P_3)| + s')/2$. So, we may assume that $s' = 0$. To solve the problem, it suffices to characterize all connected claw-free graphs G of order n for which the equality holds in the lower bound $n/2$ on $\gamma_{oir2}(G)$.

In this section, we show that the OI2RD number of a claw-free graph can be bounded from below by half of its order. In order to characterize all claw-free graphs attaining this bound, we call a graph of the following form a *k-unit* in which the number of triangles is $k - 1$.

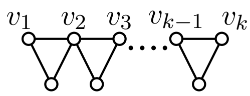

Figure 1: A k-unit.

Note that a 1-unit is isomorphic to K_1. We now let $\mathcal{G}$ be the family of all graphs of the form G_1, G_2 and G_3 depicted in Figure 2.

Theorem 2.1. *Let G be a connected claw-free graph of order n. Then, $\gamma_{oir2}(G) \ge n/2$ with equality if and only if $G \in \mathcal{G}$.*

Proof. Let f be a $\gamma_{oir2}(G)$-function. We set $Q = V_\emptyset \cap N(V_{\{1,2\}})$. Since G is a claw-free graph and because $V_\emptyset$ is independent, every vertex in $V_{\{1,2\}}$ has at most two neighbors in Q. Thus, $|Q| \le 2|V_{\{1,2\}}|$. On the other hand, every vertex in $V_\emptyset \setminus Q$ has at least two neighbors in $V_{\{1\}} \cup V_{\{2\}}$. This implies that $2|V_\emptyset \setminus Q| \le |[V_\emptyset \setminus Q, V_{\{1\}} \cup V_{\{2\}}]| \le 2(|V_{\{1\}}| + |V_{\{2\}}|)$. So, $|V_\emptyset \setminus Q| \le |V_{\{1\}}| + |V_{\{2\}}|$. We now have

$$\begin{aligned} 2\big(n - \gamma_{oir2}(G)\big) &\le 2(n - |V_{\{1\}}| - |V_{\{2\}}| - |V_{\{1,2\}}|) = 2|V_\emptyset| = 2|Q| + 2|V_\emptyset \setminus Q| \\ &\le 2(|V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|) = 2\gamma_{oir2}(G), \end{aligned} \tag{1}$$

implying the lower bound.

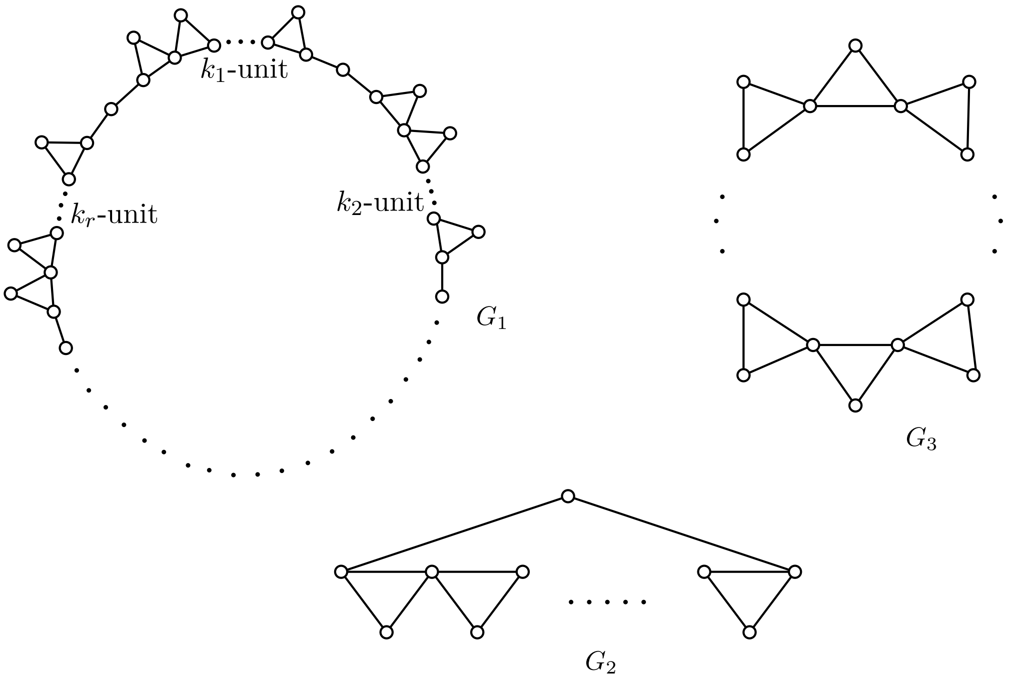

Figure 2: The claw-free graphs G_1, G_2 and G_3. In G_2 (resp. G_3), the number of triangles is odd (resp. even). In G_1, $k_1 + \cdots + k_r$ is even.

Suppose that the equality holds for a connected claw-free graph G. Then, all inequalities in (1) necessarily hold with equality. In particular, $V_{\{1,2\}} = \emptyset$ (and consequently $Q = \emptyset$) by the equality instead of the first inequality in (1). This implies that every vertex in $V_\emptyset$ has at least one neighbor in each of $V_{\{1\}}$ and $V_{\{2\}}$. Taking this fact into account, the resulting equality $2|V_\emptyset| = |[V_\emptyset, V_{\{1\}} \cup V_{\{2\}}]|$ shows that every vertex in $V_\emptyset$ has precisely one neighbor in each of $V_{\{1\}}$ and $V_{\{2\}}$. On the other hand, $|[V_\emptyset, V_{\{1\}} \cup V_{\{2\}}]| = 2(|V_{\{1\}}| + |V_{\{2\}}|)$ follows that every vertex in $V_{\{1\}} \cup V_{\{2\}}$ is adjacent to exactly two vertices in $V_\emptyset$.

Let $H = G[V_{\{1\}} \cup V_{\{2\}}]$. Suppose to the contrary that $\deg_H(v) \geq 3$ for some $v \in V(H)$. Since G is claw-free and because v has two neighbors in $V_\emptyset$, it follows that every vertex in $N_H(v)$ must be adjacent to at least one of the two neighbors of v, say x_1 and x_2, in $V_\emptyset$. This implies that $\deg(x_1) \geq 3$ or $\deg(x_2) \geq 3$, a contradiction. The above discussion guarantees that $\Delta(H) \leq 2$, and thus H is isomorphic to a disjoint union of some cycles and paths.

Let H' be a cycle $v_1 v_2 \cdots v_t v_1$ as a component of H. Let v_{11} and v_{12} be the neighbors of v_1 in $V_\emptyset$. Since G is claw-free, both v_2 and v_t have at least one neighbor in $\{v_{11}, v_{12}\}$. Furthermore, because $\deg(v_{11}) = \deg(v_{12}) = 2$, both v_2 and v_t have exactly one neighbor in $\{v_{11}, v_{12}\}$. We may assume, without loss of generality, that $v_{11}v_t, v_{12}v_2 \in E(G)$. Let v_{22} be the second neighbor of v_2 in $V_\emptyset$. Again, because G is claw-free and $v_{12}v_{22}, v_{12}v_3 \notin E(G)$, we infer that $v_{22}v_3 \in E(G)$. Iterating this process results in a graph of the form G_3, in which $v_1 v_2 \cdots v_t v_1$ is the resulting cycle by removing all vertices of degree two. In addition, since all the vertices in $V_\emptyset$ have degree two, both $\{1\}$ and $\{2\}$ must appear on their neighbors on the cycle. This implies that t is even. Note that each vertex of H' has no other neighbors in G. Now G is of the form G_3 by its connectedness.

In what follows, we may assume that H does not have any cycle as a component. If H is an edgeless graph, then G is isomorphic to the cycle C_n. Notice that

$$\gamma_{oir2}(C_p) = \lfloor p/2 \rfloor + \lceil p/4 \rceil - \lfloor p/4 \rfloor \tag{2}$$

for $p \geq 3$ (see [4]). Since $\gamma_{oir2}(C_n) = n/2$, the formula (2) shows that $n \equiv 0 \pmod 4$. It is then easy to observe that G is of the form of G_1, in which $r = n/2$ and $k_1 = \cdots = k_r = 1$. Suppose now that H is not edgeless and let H'' be a path $v_1v_2\cdots v_t$, on $t \geq 2$ vertices, as a component of H. Let v_2 be adjacent to two vertices v_{21} and v_{22} in $V_\emptyset$. Note that v_1 must be adjacent to at least one of v_{21} and v_{22}, for otherwise G would have a claw as an induced subgraph. If v_1 is adjacent to both v_{21} and v_{22}, then $G[v_1, v_2, v_{21}]$ is a 2-unit and $G \cong G[v_1, v_2, v_{21}, v_{22}] \cong K_4 - v_{21}v_{22}$. In such a case, G is of the form G_2 since it is connected. So, $G \in \mathcal{G}$. In what follows, we assume that v_1 is adjacent to only one of v_{21} and v_{22}. We then proceed with v_2. Since G is claw-free, it follows that both v_1 and v_3 must have neighbors in $\{v_{21}, v_{22}\}$. Moreover, $\deg(v_{21}) = \deg(v_{22}) = 2$ shows that both v_1 and v_3 have exactly one neighbor in $\{v_{21}, v_{22}\}$. So, we may assume that $v_1v_{21}, v_3v_{22} \in E(G)$. Similarly, v_3 has two neighbors v_{31} and v_{32} in $V_\emptyset$, in which we may assume that $v_{31} = v_{22}$. By repeating this process we obtain a t-unit on the set of vertices $K = \{v_1, v_2, \cdots, v_t, u_{12}, u_{23}, \cdots, u_{(t-1)t}\}$ in which $u_{i(i+1)}$ is adjacent to both v_i and v_{i+1}, for $1 \leq i \leq t-1$. We now consider two cases depending on $A = \big(N(v_1) \cap N(v_t)\big) \setminus \{v_2\}$.

Case 1. $A \neq \emptyset$. Let u_{1t} be in A. Because the path H'' is a component of H, $A \cap V(H) = \emptyset$. Therefore, $u_{1t} \in V_\emptyset$. Since both v_1 and v_t have exactly two neighbors in $V_\emptyset$, it follows that $A = \{u_{1t}\}$. In such a situation, the subgraph induced by $K \cup \{u_{1t}\}$ is isomorphic to G because it is connected. Notice that since all vertices in $V_\emptyset$ have degree two, both $\{1\}$ and $\{2\}$ must appear on their neighbors in the $\{v_1, \cdots, v_t\}$. This shows that t is even and hence G is of the form G_2.

Case 2. $A = \emptyset$. This implies that the subgraph induced by K is a t-unit. Then v_t is adjacent to a vertex $x' \in V_\emptyset$ and x' is adjacent to vertex $x \notin V(H) \setminus V(H'')$ (note that if $x \in V(H'')$, then $x = v_j \in \{v_1, \cdots, v_{t-1}\}$. If $j = 1$, then G is of the form G_2 and hence $G \in \mathcal{G}$. If $j \geq 2$, then v_j has at least three neighbors in $V_\emptyset$ which is impossible). Let x belong to a component H''' of H. Since H does not have a cycle as a component, it follows that H''' is a path. In such a case, the vertices of H''' belong to a $|V(H''')|$-unit by a similar fashion. Iterating this process we obtain some $|V(H_1)|, \cdots, |V(H_s)|$-units constructed as above, in which $H_1 = H''$ and s is the largest integer for which there exists such a $|V(H''_s)|$-unit. Let $H_s = w_1 \cdots w_p$ and w_p be the vertex which has only one neighbor in $V_\emptyset$ in the subgraph induced by $V(H_1) \cup \cdots \cup V(H_s)$. Similar to Case 1, there exists a vertex $u_{1p} \in V_\emptyset$ adjacent to both v_1 and w_p. On the other hand, both $\{1\}$ and $\{2\}$ must appear on the neighbors of each vertex in $V_\emptyset$. This implies that $\sum_{i=1}^{s} |V(H_i)|$ must be even. Therefore, G is of the form G_1.

In both cases above, we have concluded that $G \in \mathcal{G}$.

Conversely, let $G \in \mathcal{G}$. Suppose first that G is of the form G_2. Let $v_1 \cdots v_{2t}$ be the path on the set of vertices of degree at lest three of G_2. Then $\big(f(v_{2i-1}), f(v_{2i})\big) = (\{1\}, \{2\})$ for $1 \leq i \leq t$, and $f(v) = \emptyset$ for the other vertices defines an OI2RD function with weight half of the order. Let G be of the form G_1. Let $\{u_1, \cdots, u_{2p}\}$ be the set of vertices of degree at least three of G_1 such that $u_1 \cdots u_{k_1}$ is the path in the k_1-unit, $u_{k_1+1} \cdots u_{k_1+k_2}$ is the path in the k_2-unit, and so on. It is easy to see that $\big(g(u_{2i-1}), g(u_{2i})\big) = (\{1\}, \{2\})$ for $1 \leq i \leq p$, and $g(u) = \emptyset$ for the other vertices is an OI2RD function with weight $n/2$. Finally, we suppose that G is of the form G_3. Let $x_1x_2 \cdots x_{2q}x_1$ be the cycle on the vertices of degree four. Then the assignment $\big(g(x_{2i-1}), g(x_{2i})\big) = (\{1\}, \{2\})$ for $1 \leq i \leq q$, and $g(x) = \emptyset$ for the other vertices defines an OI2RD function of G_3 with weigh half of its order. Therefore, in all three possibilities, we have concluded that $\gamma_{oir2}(G) = n/2$. This completes the proof. □

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