Convergence in Soft Topological Spaces

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Abstract

The main purpose of this work is to introduce soft convergence in soft topological space, we introduced two types of soft convergence in soft topological space namely, soft convergence (S-convergence) of soft net and soft convergence of soft filter. Also, we investigate some properties of those concepts

Keywords: soft set, soft continuous function, soft proper function, soft net, soft filter.

1. Introduction

The concept of convergence is one of the most concepts in analysis, there is more than one way to convergence theories used in topology which leads to the same results. One of based on the notion of a net in 1922 due to Moor and Smith [3]. In 1955, Bartle R.G [1] introduced nets and filters in topology. Varol, BP. and Aygun, H. [7] presented soft space and introduced some modern notations such as convergence of sequences. Ridvan Sahin, Ahmet Kucuk [4] presented soft filters and their convergence properties. In this work we introduce two types of soft convergence net and soft convergence filter in soft topological space, we also introduce the concepts of S-compact space, S-convergence (S-cluster) of nets, also we introduce some properties of the S-proper function by notations of S-exceptional set. In this work, soft topological spaces $(K, \hat{E}, \grave{\Gamma})$ denotes (sts) which no separation axioms are assumed unless otherwise mentioned.

2. Notations and Basic Definitions

This section contained the basic definitions, propositions that are needed through this work.

Definition (2.1)[2]

Let $(K, \hat{E}, \grave{\Gamma})$ be a soft topological space over K, a soft set $(G, \hat{E})$ over K is called soft neighborhood of the soft set $(F, \hat{E})$ if there is a soft open set $(H, \hat{E})$ such that $(F, \hat{E}) \widetilde{\subseteq} (H, \hat{E}) \widetilde{\subseteq} (G, \hat{E})$. If $(F, \hat{E}) = x^{e}$, then $(G, \hat{E})$ is called a soft neighborhood of .
The soft neighborhood system of a soft element x^{e} denoted by $\mathcal{N}_s(x^{e})$.

Definition (2.2)[5]

Let $(K, \hat{E}, \grave{\Gamma})$ be a soft topological space, and let $(F, \hat{E})$ be a soft set over K, then the soft closure of $(F, \hat{E})$ which denoted by $cl(F, \hat{E})$ is the soft set, defined by $cl(F, \hat{E}) = \widetilde{\cap} \{(G, \hat{E}) : (G, \hat{E})$ is soft closed and $(F, \hat{E}) \widetilde{\subseteq} (G, \hat{E})\}$.

Proposition (2.3)[6]

1) A soft point $x^{e} \widetilde{\in} cl(F, \hat{E})$ iff for all soft open set $(G, \hat{E})$ over Kcontain x^{e}, $(F, \hat{E}) \widetilde{\cap} (G, \hat{E}) \neq \widetilde{\emptyset}$.

2) A soft set $(F,\hat{E})$ is closed iff $(F,\hat{E}) = cl(F,\hat{E})$.

Definition (2.4) [2]
Let $(K,\hat{E},\grave{\Gamma})$ be a soft topological spaces, and let $(F,\hat{E})$ be a soft set over K, a soft point x^e in K is called a soft adherent point of $(F,\hat{E})$ if $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$, for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_s(x^e)$.

Definition (2.5)[2]
Let $(K,\hat{E},\grave{\Gamma})$, $(\mathcal{M},\hat{E},\grave{\grave{\Gamma}})$ be soft topological spaces, a function $h: K \to \mathcal{M}$ is called soft continuous at $x^e \tilde{\in} \tilde{K}$ if for all soft open set $(F,\hat{E})$ containing $h(x^e)$, there is a soft open set $(G,\hat{E})$ containing x^e such that $h((G,\hat{E})) \tilde{\subseteq} (F,\hat{E})$.

Definitions (2.6)[2]
A soft topological space $(K,\hat{E},\grave{\Gamma})$, is called:
i) ST_1- space if for each $x^e, y^e \tilde{\in} \tilde{K}$ such that $x^e \neq y^e$, there are soft open sets $(F,\hat{E}),(G,\hat{E})$ such that $x^e \tilde{\in} (F,\hat{E})$, $y^e \tilde{\notin} (F,\hat{E})$ and $x^e \tilde{\notin} (G,\hat{E})$, $y^e \tilde{\in} (G,\hat{E})$.
ii) ST_2- space if for each $x^e, y^e \tilde{\in} \tilde{K}$ such that $x^e \neq y^e$, there are soft open sets $(F,\hat{E}),(G,\hat{E})$such that $x^e \tilde{\in} (F,\hat{E})$, $y^e \tilde{\in} (G,\hat{E})$ and $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) = \tilde{\emptyset}$.

Definition (2.7)[2]
A family Ω of soft sets is a cover of a soft set $(F,\hat{E})$ if $(F,\hat{E}) \tilde{\subseteq} \tilde{\cup} (G_i,\hat{E}) \tilde{\in} \Omega, i \in I\}$. It is a soft open cover if all members of Ω is a soft open set. A sub cover of Ω is a subfamily of Ω which is also a cover.

Definition (2.8)[2]
A soft topological space $(K,\hat{E},\grave{\Gamma})$ is called soft compact space if every soft open cover of $\tilde{K}$ has a finite sub cover.

Definition (2.9)[2]
A soft subspace $(\mathcal{M},\hat{E},\Gamma_{\mathcal{M}})$ of a soft topological space $(K,\hat{E},\grave{\Gamma})$, is a soft compact iff every cover of $\tilde{\mathcal{M}}$ by soft open sets in K contains a finite sub cover.

Theorem (2.10)[2]
(i) Every soft closed subset of a soft compact space is soft compact.
(ii) Every soft compact subspace of ST_2 −space is soft closed.
(iii) Soft continuous image of soft compact space is soft compact.

Definition (2.11)[2]
A function $h: (K,\hat{E},\grave{\Gamma}) \to (\mathcal{M},\hat{E},\grave{\grave{\Gamma}})$ is called:
1) Soft closed function if $h((F,\hat{E}))$ is soft closed set over $\mathcal{M}$ for every soft closed set $(F,\hat{E})$ over K.
2) Soft compact function if $h^{-1}((F,\hat{E}))$ is soft compact set over K for all soft compact set $(F,\hat{E})$ over $\mathcal{M}$.
3) Soft proper function if :
i) h is soft continuous function.
ii) The function $h \times i_Z: K \times Z \to \mathcal{M} \times Z$ is soft continuous function for all soft space Z.

Proposition (2.12)[2]
A function $h: K \to \mathcal{M}$ is soft proper if:
i) h is soft continuous function
ii) h is soft closed function
iii) $h^{-1}(\{\mathcal{y}^{e}\})$ is soft compact for each $\mathcal{y}^{e} \tilde{\in} \widetilde{\mathcal{M}}$.

3. Soft convergence of soft net
In this section, we introduced the basic definitions, theorems and remarks about soft convergence of sot net, and we give some results about them.

Definition (3.1)[2]
Let K be an ordinary set, and SP be the set of each soft point in K. The function $\tilde{\eta}: D \to SP$ is called a soft net (S-net) in K and is denoted by $\{\eta_d^{ed}\}_{d\in D}$where D is a direct set.

Definition (3.2)[2]
A S-net $\{\gamma_h^{eh}\}_{h\in H}$ in K is named a soft subnet of a S-net $\{\eta_d^{ed}\}_{d\in D}$ in K iff there is a function $\psi: H \to D$such that:
1) $\tilde{\gamma} = \tilde{\eta} \circ \psi$, i.e for all $i \in H, \gamma_i = \eta(\psi(i))$.
2) for all $d \in D$, there is $h \in H$ such that, if $p \in H, p \le h, \psi(p) \le d$.

Definition (3.3)[2]
Let $\{\eta_d^{ed}\}_{d\in D}$ be a S-net in a soft topological space $(K, \hat{E}, \grave{\Gamma})$ and $(F, \hat{E})$be a soft set over K, then:
1) $\{\eta_d^{ed}\}_{d\in D}$ is eventually in $(F, \hat{E})$ if there is $d_0 \in D$ such that $\eta_d^{ed} \tilde{\in} (F, \hat{E})$ for all $d \ge d_0$.
2) $\{\eta_d^{ed}\}_{d\in D}$ is frequently in $(F, \hat{E})$ if for all $d \in D$, there is $d_0 \in D$ with $d_0 \ge d$ such that $\eta_{d_0}^{ed_0} \tilde{\in} (F, \hat{E})$.

Remark (3.4)[2]
For all eventually S-net in K is frequently, but the converse isn't true in general.

Definitions (3.5)[2]
A S-net $\{\eta_d^{ed}\}_{d\in D}$ in a soft topological space $(K, \hat{E}, \grave{\Gamma})$ is called:
1) Converge to a soft point x^{e} (S-convergence) if it is eventually in every soft neighborhood of x^{e} (written $\eta_d^{ed} \to x^{e}$), and x^{e} is called soft limit (S-limit) point of $\{\eta_d^{ed}\}_{d\in D}$.
2) Have no S-convergent subnet in $(K, \hat{E}, \grave{\Gamma})$ (written $\eta_d^{ed} \to \infty$) iff every subnet of $\{\eta_d^{ed}\}_{d\in D}$ has no S-limit point.
3) Have a soft cluster (S-cluster) point $x^{e} \tilde{\in} \tilde{K}$ if it is frequently in every soft neighborhood of x^{e} (written $\eta_d^{ed} \alpha\, x^{e}$).

Proposition (3.6)[2]
If $\eta_d^{ed} \to x^{e}$, then $\eta_d^{ed} \alpha\, x^{e}$.

Theorem (3.7)[2]
Let $(F, \hat{E})$ be a soft set over K and $x^{e} \tilde{\in} \tilde{K}$, then $x^{e} \tilde{\in} cl(F, \hat{E})$ iff there exists a S-net $\{\eta_d^{ed}\}_{d\in D}$ in $(F, \hat{E})$ such that $\eta_d^{ed} \to x^{e}$.

Corollary (3.8)[2]

A S-net $\{\eta_d^{ed}\}_{d\in D}$ is called to have a S-cluster point $x^e \tilde{\in} \widetilde{K}$ iff $\{\eta_d^{ed}\}_{d\in D}$ have a subnet converges to x^e.

Remark (3.9)[2]
Let $h: K \to \mathcal{M}$ be a function, then:
1) If $\{\eta_d^{ed}\}_{d\in D}$ be a S-net in K, then $\{h(\eta_d^{ed})\}_{d\in D}$ is a soft net in $\mathcal{M}$.
2) If $\{\gamma_d^{ed}\}_{d\in D}$ be a S-net in $\mathcal{M}$, then there is a S-net $\{\eta_d^{ed}\}_{d\in D}$ in K such that $h(\eta_d^{ed}) = \gamma_d^{ed}$ for each $d \in D$.
3) If $\eta_d^{ed} = x^e$ for all $d \in D$, then $\eta_d^{ed} \to x^e$.

Theorem (3.10)[2]
A function $h: K \to \mathcal{M}$ is soft continuous at $x^e \tilde{\in} \widetilde{K}$ iff for all S-net $\{\eta_d^{ed}\}_{d\in D}$ in K with $\eta_d^{ed} \to x^e$, then $h(\eta_d^{ed}) \to h(x^e)$.

Theorem (3.11)[2]
A S-net $\{\omega_d^{ed}\}_{d\in D}$ in product soft topological space $\prod K_\varepsilon$, $\varepsilon \in \Omega$ is convergence to $x^e \tilde{\in} \prod K_\varepsilon$ if $P_{r\varepsilon}(\omega_d^{ed}) \to P_{r\varepsilon}\ (x^e)$ in $\widetilde{K}_\varepsilon$ for each $\varepsilon \in$, where $P_{r\varepsilon}$ is a soft projection function from $\prod K_\varepsilon$ to K_ε.

Theorem (3.12)[2]
A soft topological space $(K, \hat{E}, \grave{\Gamma})$ is ST_2- space iff every S-net has a unique S-limit point.

Proposition (3.13)
If $\{\eta_d^{ed}\}_{d\in D}$ be a S-net in converge to x^e in K, then every subnet of $\{\eta_d^{ed}\}_{d\in D}$ is converges to x^e.
The proof is clear from definition (3.3(1)) and (3.5(2)).

Theorem (3.14)
Let $\{\eta_d^{ed}\}_{d\in D}$ be a S-net in soft topological space $(K, \hat{E}, \grave{\Gamma})$ and for each $d_0 \in D$, $(F, \hat{E})_{d_0} = \{\eta_d^{ed} : d \geq d_0\}$, $x^e \tilde{\in} \widetilde{K}$ is S-cluster point of $\{\eta_d^{ed}\}_{d\in D}$ iff $x^e \tilde{\in} cl(F, \hat{E})_{d_0}$ for all $d_0 \in D$.
Proof: If x^e is S-cluster point of $\{\eta_d^{ed}\}_{d\in D}$, then for each $d_0 \in D$, $(F, \hat{E})_{d_0}$ intersect each S-neighborhood of x^e because $\{\eta_d^{ed}\}_{d\in D}$ is frequently in each S-neighborhood of x^e, then $x^e \tilde{\in} cl(F, \hat{E})_{d_0}$.
Conversely: If x^e be not S-cluster point of $\{\eta_d^{ed}\}_{d\in D}$, then there is $(G, \hat{E}) \tilde{\in} \mathcal{N}_s(x^e)$ such that $\{\eta_d^{ed}\}_{d\in D}$ isn't frequently in $(G, \hat{E})$, hence for some $d_0 \in D$ if $d \geq d_0$, then $\eta_d^{ed} \tilde{\notin} (G, \hat{E})$.
Then $(F, \hat{E})_{d_0} \tilde{\cap} (G, \hat{E}) = \widetilde{\emptyset}$, then $x^e \tilde{\in} cl(F, \hat{E})_{d_0}$.

Definition (3.15)
Let $h: (K, \hat{E}, \grave{\Gamma}) \to (\mathcal{M}, \hat{E}, \grave{\grave{\Gamma}})$ be a function, a soft set $(F, \hat{E})_h$ of $h(K)$ which is defined by $(F, \hat{E})_h = \{\gamma^e \tilde{\in} h(K)$: there is S-net $\{\eta_d^{ed}\}_{d\in D}$ in K with $\eta_d^{ed} \to \infty$ and $h(\eta_d^{ed}) \to \gamma^e\}$ is called exceptional soft set of h.

Theorem (3.16)
Let $h: (K, \hat{E}, \grave{\Gamma}) \to (\mathcal{M}, \hat{E}, \grave{\grave{\Gamma}})$ be a soft continuous function, where $(K, \hat{E}, \grave{\Gamma})$ is soft compact, and $(K, \hat{E}, \grave{\Gamma})$, $(\mathcal{M}, \hat{E}, \grave{\grave{\Gamma}})$ are $ST_2 -$ spaces. Then the following statements are equivalents:
1) h is S-proper function
2) h is soft closed function and $h^{-1}(\gamma^e)$ is soft compact set over K for all $\gamma^e \tilde{\in} \widetilde{\mathcal{M}}$.

3) If $\{\eta_d^{ed}\}_{d\in D}$ is a S-net in $(K,\hat{E},\grave{\Gamma})$ and $\gamma^e \tilde{\in} \widetilde{\mathcal{M}}$ is soft cluster point of $\{h(\eta_d^{ed})\}_{d\in D}$, then there is a S-cluster point $x^e \tilde{\in} \widetilde{K}$ of $\{\eta_d^{ed}\}_{d\in D}$ such that $h(x^e) = \gamma^e$.

Proof: $1\rightarrow 2$ Let $(F,\hat{E})$ be a soft closed set over K, since $(K,\hat{E},\grave{\Gamma})$ is soft compact, then $(F,\hat{E})$ is soft compact set over K (by theorem (2.10(i)).

Since h is soft continuous function, then $h(F,\hat{E})$ is soft compact set over $\mathcal{M}$ (by theorem 1.10(iii))

Since $(\mathcal{M},\hat{E},\grave{\Gamma})$ is a ST_2- space, then $h(F,\hat{E})$ is soft closed over $\mathcal{M}$ (by theorem 2.10(ii)), then h is soft closed function.

Now : $\{\gamma^e\}$is a soft closed set over $\mathcal{M}$ for all $\gamma^e \in \widetilde{\mathcal{M}}$, since h is soft continuous function, then $h^{-1}(\gamma^e)$ is soft closed set over K, since $(K,\hat{E},\grave{\Gamma})$ is soft compact space , then $h^{-1}(\gamma^e)$ is soft compact (by theorem 2.10(i)).

$2\rightarrow 3$ let $\{\eta_d^{ed}\}_{d\in D}$ be a S-net over K and $\gamma^e \in \widetilde{\mathcal{M}}$ be a S-cluster point of a soft net $\{h(\eta_d^{ed})\}_{d\in D}$ over $\mathcal{M}$.Claim $h^{-1}(\gamma^e) \neq \widetilde{\emptyset}$ and suppose that the statements (3) not true, that means for $x^e \tilde{\in} h^{-1}(\gamma^e)$, there is soft open set $(F,\hat{E})_{x^e}$ over K contains x^e such that $\{\eta_d^{ed}\}_{d\in D}$isn't frequently in $(F,\hat{E})_{x^e}$.

Notice that $h^{-1}(\gamma^e) = \widetilde{U}_{x^e \in h^{-1}\{\gamma^e\}}\{x^e\}$. Then the family$\{(F,\hat{E})_{x^e}: x^e \tilde{\in} h^{-1}(\gamma^e)\}$ is soft open cover of $h^{-1}(\gamma^e)$, but $h^{-1}(\gamma^e)$ is soft compact set, then there are $x_1^e, x_2^e, \dots, x_n^e$ such that $h^{-1}(\gamma^e) \tilde{\subseteq} \tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$,then $h^{-1}(\gamma^e) \tilde{\cap} (\tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e})^c = \widetilde{\emptyset}$.

Then $h^{-1}(\gamma^e) \tilde{\cap} (\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e}) = \widetilde{\emptyset}$, but $\{\eta_d^{ed}\}_{d\in D}$ isn't frequently in $(F,\hat{E})_{x_i^e}$ $for\ all\ i = 1,2,\dots,n$, thus isn't frequently in $\tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$, but $\tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$ is soft open set over K, so $\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e}$is soft closed set in K.Thus by assumption $h(\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e})$ is soft closed in $\mathcal{M}$.

Claim $\gamma^e \tilde{\notin} h(\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e})$, if $\gamma^e \tilde{\in} h(\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e})$ then there is $x^e \tilde{\in} \tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e}$ such that $h(x^e) = \gamma^e$, thus $x^e \tilde{\notin} \tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$, but $x^e \tilde{\in} h^{-1}(\gamma^e)$, therefore $h^{-1}\{\gamma^e\} \tilde{\not\subseteq} \tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$, this is contradiction, then there is soft open set $(G,\hat{E})$ over K such that

$h^{-1}(G,\hat{E}) \tilde{\cap} h^{-1}(h(\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e})) = \widetilde{\emptyset}$, i.e $h^{-1}(G,\hat{E}) \tilde{\cap} (\tilde{\cap}_{i=1}^n (F^c,\hat{E})_{x_i^e}) = \widetilde{\emptyset}$, then

$h^{-1}(G,\hat{E}) \tilde{\subseteq} \tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$, but $h(\{\eta_d^{ed}\}_{d\in D})$ is frequently in $(G,\hat{E})$, then $\{\eta_d^{ed}\}_{d\in D}$is frequently in $h^{-1}(G,\hat{E})$ and then it is frequently in$\tilde{\cup}_{i=1}^n (F,\hat{E})_{x_i^e}$.This is a contradiction, then there is S-cluster point x^e in K such that $h(x^e) = \gamma^e$.

$3\rightarrow 1$ To prove that $h \times I_Z: K \times Z \rightarrow \mathcal{M} \times Z$ is S-closed function for any soft space Z, let $(F,\hat{E})$ be soft closed over $K \times Z$and let $h \times I_Z(F,\hat{E}) = (G,\hat{E})$.

To prove that $(G,\hat{E})$ is soft closed set over $\mathcal{M} \times Z$, let $(\gamma^e, z^e) \tilde{\in} cl(G,\hat{E})$, by theorem (3.7) there is a S-net $(\gamma_d^{ed}, z_d^{ed})_{d\in D}$ in $(G,\hat{E})$ such that $(\gamma_d^{ed}, z_d^{ed})_{d\in D} \rightarrow (\gamma^e, z^e)$, thus there is a S-net $\{(\eta_d^{ed}, z_d^{ed})\}_{d\in D}$in $(F,\hat{E})$ such that $(h \times I_Z)\{(\eta_d^{ed}, z_d^{ed})\}_{d\in D} = (\gamma_d^{ed}, z_d^{ed})$ for all $d \in D$ (by theorem 3.9(2)).

Then by theorem (3.11) we have $h(\gamma_d^{ed}) \rightarrow \gamma^e$ and $I_Z(z_d^{ed}) \rightarrow z^e$and $h(x^e) = \gamma^e$.

Since$\{\eta_d^{ed}\}_{d\in D}$ is a S-net in $(F,\hat{E})$ for some soft point x^e (see definition(3.15), thus by theorem(3.11) $(\eta_d^{ed}, z_d^{ed})_{d\in D} \rightarrow (x^e, z^e)$. Since $(F,\hat{E})$ is soft closed set , then $(F,\hat{E}) = cl(F,\hat{E})$(see proposition 2.3(2)).

Then $(\gamma^e, z^e) = (h \times I_Z)(x^e, z^e) \tilde{\in} (G,\hat{E})$. Then $(G,\hat{E}) = cl(G,\hat{E})$, hence $(G,\hat{E})$ is soft closed of $\mathcal{M} \times Z$.

Then $h \times I_Z$ is soft continuous function, thus $h \times I_Z$ is S-proper function.

Theorem (3.17)

Let $h: K \to M$where $(K, \hat{E}, \grave{\Gamma})$ is $ST_2 -$ space , then h is S-proper function iff $(F, \hat{E})_h = \widetilde{\emptyset}$.
Proof: Let h be a S-proper function and suppose if possible that $(F, \hat{E})_h \neq \widetilde{\emptyset}$, then there is a soft point $\gamma^e \widetilde{\in} (F, \hat{E})_h$. Then there is a S-net $\{\eta_d^{ed}\}_{d\in D}$ over K with $\eta_d^{ed} \to \infty$ such that $h(\eta_d^{ed}) \to \gamma^e$, then there is a soft point $x^e \widetilde{\in} \widetilde{K}$ such that $\eta_d^{ed} \to x^e$ and $h(x^e) = \gamma^e$. Then we have the net $\{\eta_d^{ed}\}_{d\in D}$ is S-convergent and this contradiction, then $(F, \hat{E})_h = \widetilde{\emptyset}$.
Conversely: Let $(F, \hat{E})_h = \widetilde{\emptyset}$, to show that $(h \times I_Z): K \times Z \to \mathcal{M} \times Z$ is soft closed for any soft space Z. Let $(F, \hat{E})$ be soft closed over $K \times Z$ and $(h \times I_Z)(F, \hat{E}) = (G, \hat{E})$, to prove that $(G, \hat{E})$ is soft closed set over $\mathcal{M} \times Z$, let $(\gamma^e, z^e) \widetilde{\in} cl(G, \hat{E})$. Then by theorem(3.7) there is a S-net $(\gamma_d^{ed}, z_d^{ed})$ in $(G, \hat{E})$ such that $(\gamma_d^{ed}, z_d^{ed}) \to (\gamma^e, z^e)$, thus there is a S-net (η_d^{ed}, z_d^{ed}) such that $(h \times I_Z)(\eta_d^{ed}, z_d^{ed})_{d\in D} = (\gamma_d^{ed}, z_d^{ed})$ for all $d \in D$.
Theorem (3.11) we have $h(\eta_d^{ed}) \to \gamma^e$ and $I_Z(z_d^{ed}) \to z^e$ .Since $(F, \hat{E})_h = \widetilde{\emptyset}$, then $\eta_d^{ed} \to x^e$ for some $x^e \widetilde{\in} \widetilde{K}$, then by theorem(3.11) we gate $(\eta_d^{ed}, z_d^{ed}) \to (x^e, z^e)$. Since $(F, \hat{E})$ is soft closed set, then by theorem (3.7) we gate $(x^e, z^e) \widetilde{\in} (F, \hat{E})$.
Since $(h \times I_Z)$ is S-continuous function, then $(h \times I_Z)(\eta_d^{ed}, z_d^{ed}) = \left(h(\eta_d^{ed}), I_Z(z_d^{ed})\right) \to (h \times I_Z)(x^e, z^e) = h(x^e) \times I_Z(z^e)$. Then by theorem (3.16(3)) we have $h(x^e) = \gamma^e$ which implies to $(\gamma^e, z^e) \widetilde{\in} (G, \hat{E})$, then $(G, \hat{E})$ is soft closed set.

Theorem (3.18)[2]
The composition of two S-proper function is S-proper**.**

Theorem(2.19)
Let $f_1: K_1 \to \mathcal{M}_1$ and $f_2: K_2 \to \mathcal{M}_2$ be two S-proper functions where K_i and $\mathcal{M}_i$ are ST_1- space , i=1,2, then $f_1 \times f_2: K_1 \times K_2 \to \mathcal{M}_1 \times \mathcal{M}_2$ is S-proper function iff f_i is S-proper function.
Proof: We want to prove $(F, \hat{E})_{f_1 \times f_2} = \widetilde{\emptyset}$, if not there is $(y_1^e, y_2^e) \widetilde{\in} (F, \hat{E})_{f_1 \times f_2}$, then there is a S-net $(\eta_{d_1}^{ed_1}, \eta_{d_2}^{ed_2})$ in $K_1 \times K_2$ which has no S-limit point such that $(f_1 \times f_2)(\eta_{d_1}^{ed_1}, \eta_{d_2}^{ed_2}) \to (y_1^e, y_2^e)$, then by theorem(3.11) we have $(f_1(\eta_{d_1}^{ed_1}), f_2(\eta_{d_2}^{ed_2})) \to (y_1^e, y_2^e)$, then $f_1(\eta_{d_1}^{ed_1}) \to y_1^e$ and $f_2(\eta_{d_2}^{ed_2}) \to y_2^e$, but f_1 and f_2 are S-proper, then by theorem (3.17) we have $(F, \hat{E})_{f_1} = \widetilde{\emptyset}$and $(F, \hat{E})_{f_2} = \widetilde{\emptyset}$ which implies that $\eta_{d_1}^{ed_1} \to x^{e^1} \widetilde{\in} \widetilde{K}_1$ and $\eta_{d_2}^{ed_2} \to x^{e_2} \widetilde{\in} \widetilde{K}_2$, then $(\eta_{d_1}^{ed_1}, \eta_{d_2}^{ed_2}) \to (x^{e_1}, x^{e_2})_{f_1 \times f_2}$ this is contradiction, therefore $(F, \hat{E})_{f_1 \times f_2} = \widetilde{\emptyset}$, then $f_1 \times f_2$ is S-proper function.
Conversely: We want to prove that f_1 and f_2 are S-proper functions, since $f_1 \times f_2$ is S-proper, then by theorem (3.17) we have $(F, \hat{E})_{f_1 \times f_2} = \widetilde{\emptyset}$.
Suppose that $y_1^{e_1} \widetilde{\in} \widetilde{\mathcal{M}}_1$ and $y_1^{e_1} \widetilde{\in} (F, \hat{E})_{f_1}$, then there is a S-net $\{\eta_{d1}^{ed1}\}_{d1\in D}$ in $\widetilde{K}_1$ with $\eta_{d1}^{ed_1} \to \infty$ and $f(\eta_{d_1}^{ed_1}) \to y_1^e$, thus for each S-net $\{\eta_{d_2}^{ed_2}\}_{d2\in D}$ in $\widetilde{K}_2$, the S-net $(\eta_{d_1}^{ed_1}, \eta_{d_2}^{ed_2})$ has no S-limit point, otherwise $\{\eta_{d_1}^{ed_1}\}$ has S-limit point, if we take $\eta_{d_2}^{ed_2} = x^{e_0}\ for\ all\ d_2 \in D$, then $f_2(\eta_{d_2}^{ed_2}) \to f_2(x^{e^0}) = y_2^e$, but this implies that $(f_1 \times f_2)(\eta_{d_1}^{ed_1}, \eta_{d_2}^{ed_2}) = (f_1(\eta_{d_1}^{ed_1}), f_2(\eta_{d_2}^{ed_2})) \to (y_1^e, y_2^e)$, that is $(y_1^e, y_2^e) \widetilde{\in} (F, \hat{E})_{f_1 \times f_2}$ this is contradiction, there for $(F, \hat{E})_{f_1} = \widetilde{\emptyset}$, then f_1 is s-proper.
In similar way, we can prove that f_2 is s-proper function.

Theorem (3.20)[2]
Let $(K, \hat{E}, \grave{\Gamma})$ be ST_2 –space, and let $h: (K, \hat{E}, \grave{\Gamma}) \to \{a\}$ be a function, then h is S-proper iff $(K, \hat{E}, \grave{\Gamma})$ is soft compact where $a^e \widetilde{\notin} \widetilde{K}$.

Theorem (3.21)[2]

Let h be S-proper function from $(K, \hat{E}, \Gamma^{`})$ into $(\mathcal{M}, \hat{E}, \grave{\hat{\Gamma}})$, then $h^{-1}(H, \hat{E})$ is S-compact over K for every $(H, \hat{E})$ is S-compact over $\mathcal{M}$.

4.Convergence of Soft Filter

Some preliminaries about the soft filter are presented in this section by using soft set on an universal set and give several interesting properties, and we investigate the convergence theory of soft filter in a soft topological space.

Definition (4.1)[5]

A collection $\zeta$of non-null soft sets over K which is satisfies:
i) $\widetilde{\emptyset} \,\widetilde{\notin}\, \zeta$
ii) If $(F_1, \hat{E}), (F_2, \hat{E}) \,\widetilde{\in}\, \zeta$, then $(F_1, \hat{E}) \,\widetilde{\cap}\, (F_2, \hat{E}) \,\widetilde{\in}\, \zeta$
iii) If $(F_1, \hat{E}) \,\widetilde{\in}\, \zeta$ and $(F_1, \hat{E}) \,\widetilde{\subseteq}\, (F_2, \hat{E})$, then $(F_2, \hat{E}) \,\widetilde{\in}\, \zeta$.
is called soft filter (S-filter).

Example (4.2)[5]

Let $(K, \hat{E}, \Gamma^{`})$ be a soft topological space, a S-neighborhood $\mathcal{N}_S(x^e)$ of a soft point $x^e \,\widetilde{\in}\, \widetilde{K}$ is a S-filter, and is called the S-neighborhood filter.

Example (4.3)

Let $K = \{a, b, c\}$, $\hat{E} = \{\omega_1, \omega_2\}$, $\zeta = \{(F_1, \hat{E}), (F_2, \hat{E}), (F_3, \hat{E})\}$ where $F_1(\omega_1) = \emptyset, F_1(\omega_2) = K, F_2(\omega_1) = \{a, b\}, F_2(\omega_2) = K, F_3(\omega_1) = \{a, c\}, F_3(\omega_2) = \emptyset$.
Then ζ isn't S-filter, because $(F_1, \hat{E}), (F_3, \hat{E}) \,\widetilde{\in}\, \zeta$but $(F_1, \hat{E}) \,\widetilde{\cap}\, (F_3, \hat{E}) = \widetilde{\emptyset} \,\widetilde{\notin}\, \zeta$.

Definition (4.4)[5]

A sub collection $\zeta_0$of a S-filter ζ on K is called a S-filter base iff for all $(F, \hat{E}) \,\widetilde{\in}\, \zeta$ there is $(F_0, \hat{E}) \,\widetilde{\in}\, \zeta_0$ such that $(F_0, \hat{E}) \,\widetilde{\subseteq}\, (F, \hat{E})$.

Definition(4.5)[5]

If $\zeta_0$is a S-filter base for a S-filter ζ, then $\xi = \{(F, \hat{E}) : (F_0, \hat{E}) \,\widetilde{\subseteq}\, (F, \hat{E}) for\ some\ (F_0, \hat{E}) \,\widetilde{\in}\, \zeta_0\}$is a S-filter called filter generated by ζ_0.

Remark (4.6)[5]

A S-fiter ζ over K doesn't guarantee that $\zeta_\omega$is a S-filter on K for each $\omega \in \hat{E}$.

Definition (4.7)[5]

A S-filter ζon K is called S-convergence to x^e (written $\zeta \to x^e$), and the point x^e is called a S-limit point of ζ iff $(G, \hat{E}) \,\widetilde{\in}\, \zeta$ where $(G, \hat{E})$ is a S-neighborhood of x^e.
A soft point x^e is called cluster point (C-point) of ζ (written $\zeta \alpha x^e$ iff $(F, \hat{E}) \,\widetilde{\cap}\, (G, \hat{E}) \neq \widetilde{\emptyset}$ for all $(F, \hat{E}) \,\widetilde{\in}\, \zeta$ and $(G, \hat{E}) \,\widetilde{\in}\, \mathcal{N}_S(x^e)$.

Remark (4.8)

If a S-filter base $\zeta_0$convergence to $x^e \,\widetilde{\in}\, \widetilde{K}$, then a S-filter ξ generated by ζ_0 is also convergence to x^e, that is if $\zeta_0 \to x^e$, then $\xi \to x^e$.

Theorem (4.9)
A soft point $x^e \tilde{\in} \tilde{K}$ is S-limit point of a soft set $(F,\hat{E})$over K iff $(F,\hat{E}) - \{x^e\}$ belongs to some filter ζ such that $\zeta \to x^e$.
Proof: Suppose that x^e is a S-limit point, then $(G,\hat{E}) \tilde{\cap} (F,\hat{E}) - \{x^e\} \neq \tilde{\emptyset}$ for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$
Then ζ_0=$\{(G,\hat{E}) \tilde{\cap} \left((F,\hat{E}) - \{x^e\}\right)\}$ is a S-filter base for some S-filter ζ, then $(F,\hat{E}) - \{x^e\} \tilde{\in} \zeta$ and $\zeta \to x^e$
Conversely: If $(F,\hat{E}) - \{x^e\} \tilde{\in} \zeta$ with $\zeta \to x^e$, then $(G,\hat{E}) \tilde{\cap} \left((F,\hat{E}) - \{x^e\}\right) \neq \tilde{\emptyset}$ for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$.
Thus x^e is a S-limit point of $(F,\hat{E})$.

Theorem (4.10)
A function $h: K \to \mathcal{M}$is S-continuous iff $h(\zeta) \to h\,(x^e)$ whenever $\zeta \to x^e$.
Proof: Let $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(h(x^e))$. Since $h$is continuous, then $h^{-1}((G,\hat{E})) \tilde{\in} \mathcal{N}_S(x^e)$
Since $\zeta \to x^e$, then $h^{-1}((G,\hat{E})) \tilde{\in} \zeta$, thus $(G,\hat{E}) \tilde{\in} h(\zeta)$. Then $h(\zeta) \to h\,(x^e)$
Conversely: Suppose h be not continuous function at x^e, then there is a soft set $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(h(x^e))$ such that $h((F,\hat{E})) \tilde{\nsubseteq} (G,\hat{E})$ for all $(F,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$.
Then for any $(F,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, we can defined S-filter ζon $(F,\hat{E})$ such that $h(\zeta) \tilde{\notin} (G,\hat{E})$, but $\zeta$is a S-filter over K with $\zeta \to x^e$ this is contradiction, then h is S-continuous.

Theorem (4.11)
A S-filter ζ over K has a C-point x^e iff $x^e \tilde{\in} cl(F,\hat{E})$ for all $(F,\hat{E}) \tilde{\in} \zeta$.
Proof: $\zeta \alpha x^e \Leftrightarrow (F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$ for all $(F,\hat{E}) \tilde{\in} \zeta$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e) \Leftrightarrow x^e \tilde{\in} cl(F,\hat{E})$ for all $(F,\hat{E}) \tilde{\in} \zeta \Leftrightarrow x^e \tilde{\in} \tilde{\cap}\, cl(F,\hat{E})$.

Theorem (4.12)
If $\zeta \to x^e$, then $\zeta \alpha x^e$.
Proof: Suppose $\zeta \to x^e$, $(F,\hat{E}) \tilde{\in} \zeta$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$
Since $\zeta \to x^e$ and $(F,\hat{E}) \tilde{\in} \zeta$, then $(F,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$
Since $(F,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, thus $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$. Then $\zeta \alpha x^e$.

Theorem (4.13)
A S-filter base ζ_0 over K is S-convergence to $x^e \tilde{\in} \tilde{K}$ iff for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, there is $(F_0,\hat{E}) \tilde{\in} \zeta_0$ such that $(F_0,\hat{E}) \tilde{\subseteq} (G,\hat{E})$.
Proof: Let $\zeta \to x^e$, then a S-filter ζgenerated by ζ_0 is convergence to x^e (remark4.8).
Then $(G,\hat{E}) \tilde{\in} \zeta$ for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, (from definition 4.7), then there is $(F_0,\hat{E}) \tilde{\in} \zeta_0$ such that $(F_0,\hat{E}) \tilde{\subseteq} (G,\hat{E})$ (see definition 4.5)
Conversely: let $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then by hypotheses there is $(F_0,\hat{E}) \tilde{\in} \zeta_0$ such that $(F_0,\hat{E}) \tilde{\subseteq} (G,\hat{E})$. Since ζ_0 is a S-filter over K, then $(G,\hat{E}) \tilde{\in} \zeta_0$ (from definition (3.1(3)), then $(G,\hat{E}) \tilde{\in} \zeta_0$for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then $\zeta_0 \to x^e$ (by definition 4.7).

Theorem (4.14)
A S-filter ζ over K has a C-point x^e iff there is a S-filter ζ' finer than ζ which converges to x^e.

Proof: Suppose that $\zeta \alpha x^e$, then $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$. for all $(F,\hat{E}) \tilde{\in} \zeta$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$.
Then $\zeta_0{}' = \{(F,\hat{E}) \tilde{\cap} (G,\hat{E})\}$ is S-filter base for some S-filter ζ' which is finer than ζ and convergence to x^e.
Conversely: Since $\zeta \tilde{\subseteq} \zeta'$ and $\zeta' \to x^e$, then $(G,\hat{E}) \tilde{\in} \zeta'$for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$.
Since $\zeta \tilde{\subseteq} \zeta'$, then $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$ for all $(F,\hat{E}) \tilde{\in} \zeta$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then $\zeta \alpha x^e$.

Theorem (4.15)
Let $(F,\hat{E})$ be a soft set over K, $x^e \tilde{\in} \tilde{K}$, then $x^e \tilde{\in} cl(F,\hat{E})$ iff there is a S-filter ζover K such that $(F,\hat{E}) \tilde{\in} \zeta$ and $\zeta \to x^e$.
Proof: If $x^e \tilde{\in} cl(F,\hat{E})$, then $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$ for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then $\zeta_0 = (F,\hat{E}) \tilde{\cap} (G,\hat{E})$ is S-filter base for some S-filter ζ. The resulting S-filter contains $(F,\hat{E})$ and $\zeta \to x^e$.
Conversely: Let ζ be a S-filter such that $\zeta \to x^e$, then $\zeta \alpha x^e$ (by theorem 4.12).
Then $(F,\hat{E}) \tilde{\cap} (G,\hat{E}) \neq \tilde{\emptyset}$ for all $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$ and $(F,\hat{E}) \tilde{\in} \zeta$, then $x^e \tilde{\in} cl(F,\hat{E})$.

Corollary (4.16)
Let $(F,\hat{E})$ be a soft set over K, $x^e \tilde{\in} \tilde{K}$, then $x^e \tilde{\in} cl(F,\hat{E})$ iff there is a S-filter base ζ_0 over K such that $(F,\hat{E}) \tilde{\in} \zeta_0$ and $\zeta_0 \to x^e$.

Definition (4.17)
Let ζ_0 be a S-filter base over K for all $(F_1,\hat{E}),(F_2,\hat{E}) \tilde{\in} \zeta_0$, we put $(F_1,\hat{E}) \geq (F_2,\hat{E})$ iff $(F_1,\hat{E}) \tilde{\subseteq} (F_2,\hat{E})$, then $(\zeta_0,\geq)$ is directed set. For all $(F,\hat{E}) \tilde{\in} \zeta_0$ define $\eta: \zeta_0 \to \tilde{\cup} (F,\hat{E})$, $(F,\hat{E}) \tilde{\in} \zeta_0$ such that for all $(F,\hat{E}) \tilde{\in} \zeta_0$ take (fixed) $\eta_{(F,\hat{E})} \tilde{\in} (F,\hat{E})$ such that $\eta(F,\hat{E}) = \eta_{(F,\hat{E})}$
Thus $(\eta_{(F,\hat{E})})_{(F,\hat{E})} \tilde{\in} \zeta_0$ is a S-net over K and it is called a S-net associated with a S-filter base ζ_0.

Theorem (4.18)
Let $(\eta_{(F,\hat{E})})_{(F,\hat{E})} \tilde{\in} \zeta_0$be a S-net associated with a S-filter base ζ_0 on a sts $(K,\hat{E},\grave{\Gamma})$ and $x^e \tilde{\in} \tilde{K}$. If $\zeta_0 \to x^e$, then $\eta_{(F,\hat{E})} \to x^e$.
Proof: Let $\zeta_0 \to x^e$ and $(G,\hat{E}) \tilde{\in} \mathcal{N}_S(x^e)$, then there is $(F_0,\hat{E}) \tilde{\in} \zeta_0$ such that $(F_0,\hat{E}) \tilde{\subseteq} (G,\hat{E})$.
Then $\eta_{(F_0,\hat{E})} \tilde{\in} (G,\hat{E})$, So $\eta_{(F,\hat{E})} \tilde{\in} (G,\hat{E})$, for all $(F,\hat{E}) \geq (F_0,\hat{E})$, then $\eta_{(F,\hat{E})} \to x^e$.

Theorem (4.19)[5]
A sts $(K,\hat{E},\grave{\Gamma})$ is ST_2 −space iff every convergence S-filter over K has a unique S-limit point.

Theorem (4.20)
A sts $(K,\hat{E},\grave{\Gamma})$ is S-compact iff each S-filter base ζ_0 with S-adherent point x^econvergence to x^e.
Proof: Suppose that $(K,\hat{E},\grave{\Gamma})$ be a S-compact and x^e is a S-adherent point of ζ_0, then $x^e \tilde{\in} cl(F,\hat{E})$ for all $(F,\hat{E}) \tilde{\in} \zeta_0$, then $\zeta_0 \to x^e$ (see corollary 4.16)
Conversely: Suppose that $\zeta_0 \to x^e$, then by theorem (4.13) every S-net associated with a S-filter base convergence to x^e.
Since every S-net has a subnet which convergence to x^e.
Thus $(K,\hat{E},\grave{\Gamma})$ is S-compact space.

Theorem (4.21)
A S-filter on a space $\prod K_\varepsilon$, $\varepsilon \in \Omega$ is convergence to $x^e \tilde{\in} \prod K_\varepsilon$ if $P_{r\varepsilon}(\zeta) \to P_{r\varepsilon}(x^e)$ in $K_\varepsilon$for each $\varepsilon \in \Omega$.

Proof: If $\zeta \to x^e$ in $\prod K_\varepsilon$ for all $\varepsilon \in \Omega$. Since $P_{r\varepsilon}$ are soft continuous functions, then by theorem (4.10) we have $P_{r\varepsilon}(\zeta) \to P_{r\varepsilon}(x^e)$ in $\prod K_\varepsilon$ for all $\varepsilon \in \Omega$.

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