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Numerical solutions of Fokker–Planck Equation based on Hermite functions

Manoochehr Khasi[1]
Iran University of Science and Technology, Tehran, Iran

Fereshteh Samadi
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697, Tehran, Iran

Abstract

In this research study, a spectral method based on Hermite functions is considered to solve the Fokker–Planck equation which is an equation governed by the time evolution of probability density function of the stochastic processes. Spatially the equation is discretized using a Galerkin method based on Hermite functions and the system of a first–order ordinary differential equation is obtained. Using an eigenvalue decomposition technique, the equation is transformed into a set of independent ordinary differential equations that can be solved analytically. Some numerical examples are included to show the accuracy and efficiency of the proposed approach.

Keywords: Fokker–Planck equation, Hermite functions, spectral Galerkin method

AMS Mathematical Subject Classification [2010]: 65M70 , 35Q84

1 Introduction

Fokker-Planck equations can be considered as the distribution function of the position or velocity of a particle evolving with time after being subjected to a random force in the potential energy field. As far as we know, the analytical solutions of the Fokker–Planck equations can be derived only in some special cases, and generally it is hard to find them. Therefore, the computational methods play a main role in computation of the Fokker–Planck equations. Of the many computational approaches that have been studied for solving these models, we can mention the finite difference schemes [3], combined Hermite spectral-finite difference method [1] and domain decomposition spectral method [2].

In this paper, we want to solve the following Fokker–Planck Equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu + f(\mathbf{x}, t), & t > 0,\ \mathbf{x} \in \Omega, \\ u(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

[1] speaker

where $\Omega = \mathbb{R}^d$. The boundary conditions are specified as

$$u(\mathbf{x}, t) = 0, \qquad \text{on} \quad \partial\Omega.$$

The classical Fokker–Planck equation was introduced by Fokker and Planck in 1914 when they studied the Brownian motion of particles [4]. Today, this equation is the skeleton of statistical physics.

2 Proposed approach for solving the Fokker–Planck Equation

Here, we want to solve the following one-dimensional fractional Fokker–Planck Equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial u}{\partial x} + cu + f(x,t), & t \in [0,T],\ x \in \mathbb{R}, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

using Galerkin method based on Hermite functions. Suppose that $H_n(x)$ is the Hermite polynomial of degree n. The Hermite functions are defined as [5]

$$\varphi_n(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{2^n n!}} \mathrm{e}^{-\frac{-x^2}{2}} H_n(x).$$

Also, The Hermite functions can be shown by the following recursion relation

$$\begin{cases} \varphi_{n+1}(x) = \sqrt{\dfrac{2}{n+1}}\, x\varphi_n(x) - \sqrt{\dfrac{n}{n+1}}\,\varphi_{n-1}(x), & n \geqslant 1, \\ \varphi_0(x) = \pi^{-1/4}\mathrm{e}^{\frac{-x^2}{2}}, & \varphi_1(x) = \pi^{-1/4}\sqrt{2}x\mathrm{e}^{\frac{-x^2}{2}}. \end{cases}$$

Based on the orthogonality of the Hermite functions

$$\langle \varphi_n, \varphi_m \rangle := \int_{\mathbb{R}} \varphi_n(x)\varphi_m(x)\mathrm{d}x = \delta_{mn}, \qquad m, n \geqslant 0, \tag{3}$$

where δ_{mn} is the Kronecker function. It is easy to obtain the first derivative of the Hermite functions as

$$\varphi_n'(x) = \sqrt{\frac{n}{2}}\varphi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\varphi_{n+1}(x). \tag{4}$$

Using (4), we can show that

$$\langle \varphi_n', \varphi_m \rangle = \int_{\mathbb{R}} \varphi_n'(x)\varphi_m(x)\mathrm{d}x = \begin{cases} \sqrt{\dfrac{n}{2}} & m = n-1, \\ -\sqrt{\dfrac{n+1}{2}} & m = n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Also, using (4) and integration by parts Techniques

$$\langle \varphi_n'', \varphi_m \rangle = \int_{\mathbb{R}} \varphi_n''(x)\varphi_m(x)\mathrm{d}x = -\int_{\mathbb{R}} \varphi_n'(x)\varphi_m'(x)\mathrm{d}x = \begin{cases} \dfrac{\sqrt{n(n-1)}}{2} & m = n-2, \\ -\left(n+\dfrac{1}{2}\right) & n = m, \\ \dfrac{\sqrt{(n+1)(n+2)}}{2} & m = n+2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To approximate integrals based on Hermite functions, it is necessary to modify the Hermite-Gauss quadrature.

Theorem 2.1. Let $\{X_j\}_1^N$ be the zeros of Hermite polynomial of $H_N(x)$. Then

$$\int_{\mathbb{R}} \varphi_n(x) f(x) \mathrm{d}x \simeq \frac{1}{N} \sum_{j=1}^{N} \frac{\varphi_n(X_j)}{\varphi_{N-1}^2(X_j)} f(X_j), \qquad n \geqslant 0,$$

Proof. Based on the Hermite-Gauss quadrature, we have

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} f(x) \mathrm{d}x \simeq \sum_{j=1}^{N} w_j f(X_j), \qquad n \geqslant 0,$$

where

$$w_j = \frac{\sqrt{\pi} 2^{N-1} (N-1)!}{N H_{N-1}^2(X_j)} = \frac{1}{N} \frac{\mathrm{e}^{-X_j^2}}{\left(\frac{\mathrm{e}^{-\frac{X_j^2}{2}} H_{N-1}^2(X_j)}{\sqrt[4]{\pi} \sqrt{2^{N-1}(N-1)!}} \right)^2} = \frac{1}{N} \frac{\mathrm{e}^{-X_j^2}}{\varphi_{N-1}^2(X_j)},$$

therefore

$$\begin{aligned} \int_{\mathbb{R}} \varphi_n(x) f(x) \mathrm{d}x &= \int_{\mathbb{R}} \mathrm{e}^{-x^2} \varphi_n(x) f(x) \mathrm{e}^{x^2} \mathrm{d}x \simeq \sum_{j=1}^{N} w_j \varphi_n(X_j) f(X_j) \mathrm{e}^{X_j^2} \\ &= \frac{1}{N} \sum_{j=1}^{N} \frac{\varphi_n(X_j)}{\varphi_{N-1}^2(X_j)} f(X_j), \end{aligned}$$

□

Now, suppose that the approximate solution of (2) is

$$z(x,t) = \sum_{n=1}^{N} \lambda_n(t) \varphi_n(x) = \underbrace{\begin{bmatrix} \varphi_1(x) & \cdots & \varphi_n(x) \end{bmatrix}}_{\boldsymbol{\varphi}^T(x)} \underbrace{\begin{bmatrix} \lambda_1(t) \\ \vdots \\ \lambda_n(t) \end{bmatrix}}_{\boldsymbol{\lambda}(t)}. \tag{7}$$

The unknown functions $\lambda_n(t)$ will be determined from the following variational formulation

$$\begin{cases} \displaystyle\sum_{n=1}^{N} \frac{\partial \lambda_n(t)}{\partial t} \langle \varphi_n, \varphi_m \rangle = \sum_{n=1}^{N} \lambda_n(t) \langle a\ \varphi_n'' + b\ \varphi_n' + c\ \varphi_n, \varphi_m \rangle + \langle f, \varphi_m \rangle, \\ \displaystyle\sum_{n=1}^{N} \lambda_n(0) \langle \varphi_n, \varphi_m \rangle = \langle g, \varphi_m \rangle, \end{cases}$$

where

$$\langle f, \varphi_m \rangle := f_m(t) \simeq \frac{1}{N} \sum_{j=1}^{N} \frac{\varphi_m(X_j)}{\varphi_{N-1}^2(X_j)} f(X_j, t),$$

and

$$\langle g, \varphi_m \rangle := g_m \simeq \frac{1}{N} \sum_{j=1}^{N} \frac{\varphi_m(X_j)}{\varphi_{N-1}^2(X_j)} g(X_j).$$

By using (3), we obtain

$$\begin{cases} \dfrac{\partial \lambda_m(t)}{\partial t} = \displaystyle\sum_{n=1}^{N} \lambda_n(t) \underbrace{\langle a\ \varphi_n'' + b\ \varphi_n' + c\ \varphi_n, \varphi_m \rangle}_{a_{mn}} + f_m(t), \\ \lambda_m(0) = g_m, \end{cases} \tag{8}$$

Also, by using (3), (5) and (6), it can be seen that

$$a_{mn} = a\langle \varphi_{n-1}'', \varphi_{m-1}\rangle + b\langle \varphi_{n-1}', \varphi_{m-1}\rangle + c\ \langle \varphi_{n-1}, \varphi_{m-1}\rangle = \begin{cases} a\dfrac{\sqrt{(n-1)(n-2)}}{2} & m = n-2, \\ b\sqrt{\dfrac{n-1}{2}} & m = n-1, \\ -a\left(n-\dfrac{1}{2}\right) + c & n = m, \\ -b\sqrt{\dfrac{n}{2}} & m = n+1, \\ a\dfrac{\sqrt{n(n+1)}}{2} & m = n+2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the equation (8) can be represented in matrix form as

$$\frac{\partial \boldsymbol{\lambda}(t)}{\partial t} = \mathbf{A}\boldsymbol{\lambda}(t) + \mathbf{F}(t), \qquad \boldsymbol{\lambda}(0) = \mathbf{G},$$

where $\mathbf{F}(t) = [f_1(t), \cdots, f_N(t)]^T$, and other matrix and vectors can be defined similarly.

By supposing that $\mathbf{A}$ is a diagonalizable matrix, we have $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{P}$ is a square matrix whose columns are the eigenvectors of $\mathbf{A}$, and $\mathbf{D} = \mathrm{diag}(\xi_1, \xi_2, \cdots, \xi_N)$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues of $\mathbf{A}$. We should mention that when $b = 0$, the matrix $\mathbf{A}$ becomes symmetric matrix, and so $\mathbf{P}$ becomes an orthogonal matrix. By using such decompositions, the above equation becomes

$$\frac{\partial \boldsymbol{\lambda}(t)}{\partial t} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\boldsymbol{\lambda}(t) + \mathbf{F}(t), \qquad \boldsymbol{\lambda}(0) = \mathbf{G},$$

so

$$\mathbf{P}^{-1}\frac{\partial \boldsymbol{\lambda}(t)}{\partial t} = \mathbf{D}\mathbf{P}^{-1}\boldsymbol{\lambda}(t) + \mathbf{P}^{-1}\mathbf{F}(t), \qquad \mathbf{P}^{-1}\boldsymbol{\lambda}(0) = \mathbf{P}^{-1}\mathbf{G}.$$

By supposing $\boldsymbol{\mu}(t) = \mathbf{P}^{-1}\boldsymbol{\lambda}(t)$, $\widetilde{\mathbf{G}} = \mathbf{P}^{-1}\mathbf{G}$, and $\widetilde{\mathbf{F}}(t) = \mathbf{P}^{-1}\mathbf{F}(t)$, we have

$$\frac{\partial \boldsymbol{\mu}(t)}{\partial t} = \mathbf{D}\boldsymbol{\mu}(t) + \widetilde{\mathbf{F}}(t), \qquad \boldsymbol{\mu}(0) = \widetilde{\mathbf{G}}.$$

or

$$\frac{\partial \mu_i(t)}{\partial t} = \xi_i\ \mu_i(t) + \widetilde{f_i}(t), \qquad \mu_i(0) = \widetilde{g_i}, \qquad i = 1, 2, \cdots, N. \tag{9}$$

For each i, the above equation can be analytically solved as follow

$$\mu_i(t) = \widetilde{g_i}\mathrm{e}^{\xi_i t} + \int_0^t \widetilde{f_i}(s)\mathrm{e}^{\xi_i(t-s)}\mathrm{d}z.$$

Also, we can easily show that

$$\mu_i(t_{n+1}) = \mu_i(t_n)e^{\xi_i h} + \int_{t_n}^{t_{n+1}} \widetilde{f}_i(s)e^{\xi_i(t_{n+1}-s)}dz. \tag{10}$$

The above integrals can be approximated by the Clenshaw–Curtis quadrature formula or Gauss-Legendre quadrature.

3 Numerical Experiments

In this section, we demonstrate the efficiency of the proposed approach by presenting some numerical examples to solve the fractional Fokker–Planck equation. The numerical results are performed in Matlab 2015a. Also, we should mention that the integral in the formula (10) are approximated by the Clenshaw–Curtis quadrature formula with 100 grid points.

Example 3.1. Consider the following Fokker–Planck equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + (2t - 4x^2t^2 + 2t^2)\exp(-x^2), & t \in [0,1],\ x \in \mathbb{R}, \\ u(x,0) = 0, & x \in \mathbb{R}, \end{cases}$$

which the exact solution is $u(x,t) = t^2\exp(-x^2)$. In Fig. 1, the maximum absolute error with respect to N is shown. After spatial discretization of the equation, the obtained system is converted to a system of first–order ODE which can be solved analytically. Therefore, the arising errors are because of the spatial discretization and quadrature errors which both of them are spectral orders. It can be seen from Fig. 1 that the spectral convergence is almost achieved and the error for an even N is a little less than that for an odd N. Also, in Table 1, the results are shown for different values of N.

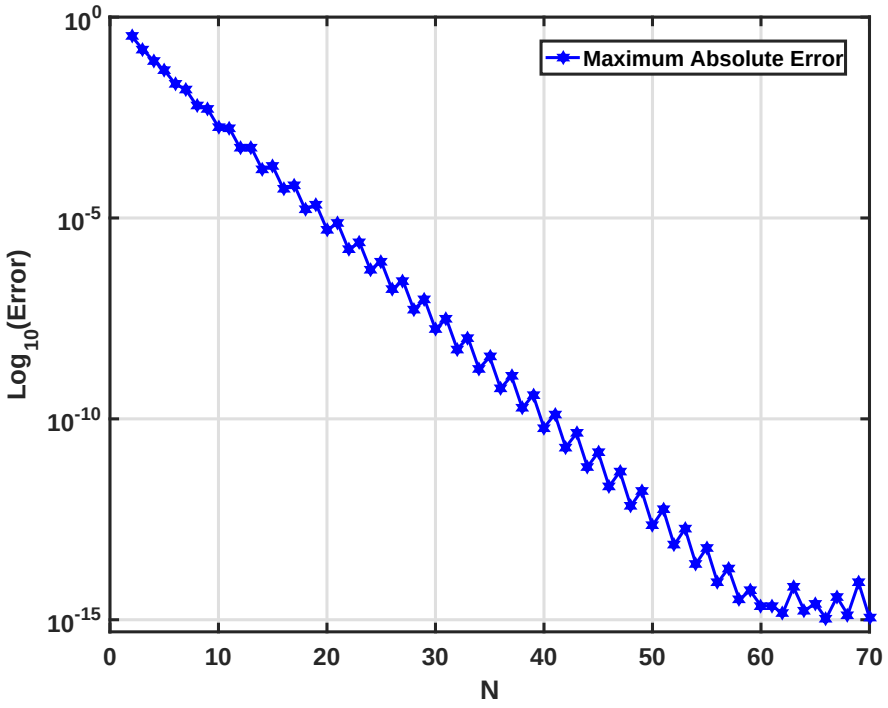

Figure 1: The maximum absolute error as a function of N for Example 3.1.

N	maximum absolute error	CPU time(s)
10	1.8190×10^{-03}	0.1019
20	5.1172×10^{-06}	0.1798
30	1.6990×10^{-08}	0.3153
40	6.0188×10^{-11}	0.4741
50	2.2171×10^{-13}	0.7342
60	2.2204×10^{-15}	0.9831

Table 1: The maximum absolute errors and running times for some values of N.

4 Conclusion

In this research study, a spectral method based on Hermite functions was considered to solve the Fokker–Planck equation which is an equation governed by the time evolution of probability density function of the stochastic processes. Spatially the equation was discretized using a Galerkin method based on Hermite functions and the system of a first–order ordinary differential equation was obtained. Using an eigenvalue decomposition technique, the equation was transformed into a set of independent ordinary differential equations that can be solved analytically. Some numerical examples were included to show the accuracy and efficiency of the proposed approach.

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e-mail: mkhasi81math@gmail.com
e-mail: f_samadi@pnu.ac.ir