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# Minimum fundamental cycle basis of graphs

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**Abstract**

A cycle basis of a simple graph $G$ is a basis for the null-space of the incidence matrix of $G$, each of whose elements corresponds to a cycle of $G$. A minimum cycle basis is a cycle basis with minimum total length. One of the important types of cycle basis is fundamental cycle basis. In this paper we compute the minimum fundamental cycle basis for special families of graph products.

**Keywords:** Cycles in graphs, Cycle basis, Minimum cycle basis.

**AMS Mathematical Subject Classification [2010]:** 05C50, 05C90

## 1 Introduction

A cycle of a graph $G = (V, E)$ is a connected regular subgraph of degree 2. Any cycle in $G$ can be represented by an incidence vector $\gamma_C \in \{0,1\}^{|E|}$ ($\gamma_C \in \{0, \pm 1\}^{|E|}$ in directed case). The *cycle space* of $G$ is the vector space generated by $\{\gamma_C | \ C \text{ is a cycle in } G\}$ over $\mathbb{Z}_2$ (over $\mathbb{Q}$ in directed case). A *cycle basis* for $G$ consists of some cycles which form a basis for cycle space of $G$. The *length of a cycle basis* is the total length of the cycles included in the basis. A *minimum cycle basis* (or MCB for short) of a graph is a cycle basis with minimum length. We associate a matrix $M(B)$ to each cycle basis $B$ in which all whose rows are the members of the basis. In [5] the authors give a good survey on cycle basis of graphs. In [7] five different classes of cycle bases are defined. One of these classes of cycle basis is strictly fundamental cycle basis (or fundamental cycle basis for short). A cycle basis $B$ is called fundamental if there exists a matrix $M'$ such that $M(B)$ equals to $[I|M']$ (after possibly rearrangement of the rows and column), where $I$ is the identity matrix. In the following Lemma we find a characterization of the fundamental basis of graphs.

**Lemma 1.1.** *Let $G$ be an undirected graph and $B$ be a cycle basis for $G$. Then $B$ is fundamental cycle basis if and only if there exits some spanning tree $T$ of $G$ such that $\mathcal{B} = \{C_T(e) | e \in E(G) \setminus E(T)\}$, where $C_T(e)$ denoted the unique cycle in $T \cup \{e\}$.*

*Proof.* We prove one part, the proof of the other part is similar. Suppose that there exits some spanning tree $T$ of $G$ such that $\mathcal{B} = \{C_T(e) | e \in E(G) \setminus E(T)\}$, where $C_T(e)$ denoted the unique cycle in $T \cup \{e\}$. We give an ordering to the set of vertices and to the set of the basis, with respect to which $M(B)$ is of the forme

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$[I|M']$. Let $e_1, e_2, \ldots, e_{n-m+1}$ be the the non-tree edge of $V$. Then in the ordering we put $e_1, e_2, \ldots, e_{n-m+1}$ first and the edges of $T$ last in $V$. Moreover let $C_T(e_1), C_T(e_2), \ldots, C_T(e_{n-m+1})$ be the ordering of the members of $B$. Then it is easy to see that by this ordering $M(B)$ has the desired standard form. □

It has been shown that computing minimum fundamental cycle basis is an NP-hard problem. Hence it would be interesting to compute the fundamental basis for special families of graphs.

## 2 Minimum Fundamental Basis

Let $K_{n,n}$ be a complete bipartite graph of order $2n$ and $M$ be a complete matching of $K_{n,n}$. Suppose $G_n$ be a graph constructed from $K_{n,n} \setminus M$. Then for each $n$ we will find a minimum weakly fundamental basis for graph $G_n$. The proof is mostly based on work in [3].

In this section we find a minimum fundamental cycle basis for $G_n$. Since $G_n$ is a bipartite graph the length of its cycles are at least 4. In the following theorem we show that an MFCB for $G_n$ should have a cycle of length 6.

**Theorem 2.1.** *for any integer $n \geq 3$, graph $G_n$ has no fundamental cycle basis with all cycles of length 4.*

*Proof.* For the contrary suppose there exists a spanning tree $T$ of $G_n$ and $\mathcal{B}$ is a fundamental cycle basis with respect to $T$ and all cycles in $\mathcal{B}$ have length 4.
First we show that there is no pair of vertices $\alpha$, $\beta$ such that $d_T(\alpha, \beta) \geq 5$. without loss of generality we suppose that $\alpha$ is a vertex of degree one in $T$. Let $P$ be the path in $T$ connecting $\alpha$ and $\beta$ and $f$ be a vertex in $P$ with $d_T(\alpha, f) = 5$. Let $Q = \alpha, b, c, d, e, f$ be the path in $T$ connecting $\alpha$ and $f$. without loss of generality assume that $\alpha \in A$, then $\{c, e\} \subseteq A$ and $\{b, d, f\} \subseteq B$. We have $\{\alpha, f\} \notin E(G_n)$, otherwise there would be a cycle of length 6 in $\mathcal{B}$, which is a contradiction. Since the only vertex of $B$ which is not adjacent to $\alpha$ is $n + \alpha$, we have $f = n + \alpha$. We claim that $N_T(e) = \{f, d\}$, because if $e$ has another neighbor called $s$ then $s \in B$ on the other hand with the same reason as before $\{\alpha, s\} \notin E(G_n)$ and hence $s = n + \alpha$, but we had $f = n + \alpha$, which is a contradiction. With a same procedure one can prove $N_T(b) = \{\alpha, c\}$. It is easy to check that if $c$ or $d$ has a neighbor which is not $Q$, then in either cases it should be a vertex of degree one of $T$. Now, we consider the vertex $n + c \in B$ the only possibility for it is that it is out of $Q$ with $d_T(n + c, f) \geq 2$. Hence, $d_T(n + c, c) \geq 5$ and $d_T(n + c, a) \geq 7$. Now adding the edge $\{a, n + c\}$, which had been already an edge of $G_n$, to $T$ gives a cycle with length at least 8 in $\mathcal{B}$, which is a contradiction.

Now, we prove that there are no pair $a$ and $e$ of vertices such that $d_T(a, e) = 4$. Without loss of generality we suppose that $a$ is a vertex of degree one in $T$. Let $P = a, b, c, d, e$ be the unique path in $T$ connecting $a$ and $e$. We claim that for every vertex $x$ of $G_n$, $d_T(c, x) \leq 2$. For contrary, suppose there exist a vertex $z \in V(G_n)$ with $d_T(c, z) = 3$. Let $Q$ be the unique path in $T$ connecting $c$ and $z$, there are three cases:
**Case 1.** $E(P) \cap E(Q) = \emptyset$. Then $d_T(a, z) = d_T(a, z) = 5$, which is a contradiction.
**Case 2.** $|E(P) \cap E(Q)| = 1$. Without loss of generality assume that $|E(P) \cap E(Q)| = \{b, c\}$. One can easily check that $d_T(z, e) = 5$, which is a contradiction.
**Case 3.** $|E(P) \cap E(Q)| = 2$. Without loss of generality assume that $|E(P) \cap E(Q)| = \{\{b, c\}\{a, b\}\}$. Again, it is easy to see that $d_T(z, e) = 5$, which is a contradiction. □

**Theorem 2.2.** *For any integer $n \geq 3$, a minimum fundamental cycle basis for graph $G_n$ consists of one cycle of length 6 and all other cycles of length 4.*

*Proof.* It is a straight conclusion of Theorem 2.1. □

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