8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

# 2-Generalized burning number in square graphs

Abolfazl Bahmani[1]
University of Zanjan, Zanjan, Iran

Mojgan Emami
University of Zanjan, Zanjan, Iran

Ozra Naserian
Department of Mathematics, Zanjan Branch, **Islamic Azad university**, Zanjan, Iran

**Abstract**

The burning number of a graph $G$, denoted by $b(G)$, is the minimum number of steps it takes to burn the graph. 2-Generalized burning number of $G$ which is a generalization of $b(G)$, denoted by $b_2(G)$, is the minimum number of steps it takes to burn every vertex of $G$ by burning vertices only if they are adjacent to at least 2 burned neighbors. In this paper, we give some bounds for the 2-generalized burning number in square graphs. Then we obtain the 2-generalized burning number in square graphs of some specific graphs .

**Keywords:** 2-generalized burning number, Square graph.

**AMS Mathematical Subject Classification [2010]:** 13D45, 39B42

## 1 Introduction

A graph $G$ is a pair $G = (V, E)$, where $V$ is the set of vertices of $G$ and $E$ is a subset of $P_2(V)$ called the edge set of $G$. We say that two vertices $u$ and $v$ in $G$ are adjacent if $(u, v) \in E$. The degree of a vertex $v$ in $G$, denoted by $\deg(v)$, is the number of vertices adjacent to $v$. A graph $G$ is $r$- regular if every vertex of $G$ has degree $r$. A path is a sequence of vertices, such that there is an edge from each vertex to the next vertex of this sequence. A path graph $P_n$ is a graph whose vertices can be listed in the order $\{v_1, v_2, , v_n\}$ such that the edges are $e_i = (v_i, v_{i+1})$, where $i = 1, 2, \cdots, n-1$. A Hamiltonian path is a path in a graph that visits each vertex exactly once. The distance between two vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is then length of the shortest path between $u$ and $v$. The eccentricity of a vertex $v$ is defined as $max\{d_G(v, u) : u \in V(G)\}$. Radius and diameter of $G$ is the minimum and maximum eccentricity over the set of all vertices of $G$, denoted by $r(G)$ and $d(G)$, respectively. A subgraph $H$ of a graph $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph $H$ of $G$ is a graph such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A complete graph is a graph in which each pair of vertices is connected by an edge. A complete graph with $n$ vertices is denoted by $K_n$. For classical terminologies, one is referring to [6].

[1]speaker

A simple graph with one vertex of degree $n-1$ and $n-1$ vertices of degree 1 is called a star graph denoted by $S_n$. A strongly regular graph with parameters $(n, k, \lambda, \mu)$ (for short, a srg$(n, k, \lambda, \mu)$) is a $k$-regular graph on $n$ vertices such that:

1. any two adjacent vertices have exactly $\lambda$ common neighbours,
2. any two nonadjacent vertices have exactly $\mu$ common neighbours[6].

The square graph $G$, denoted by $G^2$, is a graph with vertex set $V(G)$ such that two vertices $u$ and $v$ are adjacent if and only if $d_G(u, v) \leq 2$.

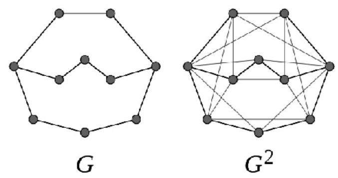


Figure 1: The square of $G$

A set $S \subseteq V(G)$ is called a $2-$dominating set if every vertex not is $S$ has at least two neighbors in $S$. Let $G$ be a connected graph, the process of burning $G$ begins with all vertices being unburned. In the first step, a single vertex is selected to be burned. In every subsequent time step, either the fire spreads to all neighbours of a previously burned vertex, and those vertices become burned, or another vertex is selected to be burned. This means that if a vertex is already burned at a step, then its unburned neighbors (if any) become automatically burned in next step. When all the vertices are burned, the process ends. The minimum number of time steps needed to burn all the vertices of a graph is named the burning number of $G$ and denoted by $b(G)$ [2]. $2-$generalized burning number of $G$, denoted by $b_2(G)$, is the minimum number of steps it takes to burn every vertex of by burning vertices only if they are adjacent to at least 2 burned neighbors. we call this discrete time process as a 2-burning process of $G$. If all vertices of the graph are burned after $k$ times steps by 2-burning process, we call sequence $(x_1, x_2, \ldots, x_k)$ a 2-burning sequence of graph. Clearly, the generalized burning number of is the length of a minimum burning sequence among all 2-burning process of a graph $G$ [5].

First we remind some theorems that we need in this study:

**Theorem 1.1.** *[5] If G is a path graph of order n, or is a circle graph of order $n \geq 3$, then:*

$$b_2(G) = \lceil \tfrac{n}{2} \rceil + 1.$$

**Theorem 1.2.** *[5] Suppose G is a complete graph of order $n \geq 2$, then:*

$$b_2(G) = 3.$$

## 2-generalized burning number in square graph

In what followes all graphs are simple and connected. We give some bounds for the 2-generalized burning number in square graphs, and obtain the 2-generalized burning number in the square graph of some specific graphs.

**Lemma 1.3.** *If $G$ is a connected graph, then we have $b_2(G) \leq \mid S \mid +1$ that $S$ is a $2-$ dominating set for $G$.*

*Proof.* Suppose that the set $S = \{a_1, a_2, \ldots, a_s\}$ is a $2-$ dominating set for $G$. Firstly, by burning the $S$ and then by selecting an unburned vertex outside $S$ , all the vertices of the graph $G$ are burned. So we have $b_2(G) \leq \mid S \mid +1$.

□

**Theorem 1.4.** *If $G$ be a graph of order $n$ that it has a hamiltonian path, then we have $b_2(G^2) \leq [\frac{n}{3}] + 2$.*

*Proof.* Suppose $S = \{v_1, v_2, \ldots, v_n\}$ be a hamiltonian path in $G$. If $n = 3k$, the set $S^{'} = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k}\}$ and If $n = 3k+1$, the set $S^{'} = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k+1}\}$ is a 2-dominating set for $G^2$. Also if $n = 3k+2$, the set $S^{'} = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k+1}, \}$ is a 2-dominating set for $G^2 - (v_{3k+2})$. Therefore, according to lemma 1.3, first by burning $S^{'}$ and then by selecting an unburned vertex outside of $S^{'}$, the graph $G^2$ is burned . So we have $b_2(G^2) \leq \mid S^{'} \mid +1 = [\frac{n}{3}] + 2$.

**Theorem 1.5.** *If $G$ is a connected graph, then we have $b_2(G^2) \leq b_2(G)$.*

*Proof.* Suppose that $G$ is a connected graph. Clearly that $G$ is an spanning subgraph of $G^2$ and $\mid E(G) \mid \leq \mid E(G^2) \mid$. Thus, we have $b_2(G^2) \leq b_2(G)$.

□

**Theorem 1.6.** *If $n = 3k + i$ where $0 \leq i \leq 2$, then $b_2({P_n}^2) = k + 2$.*

*Proof.* Since ${P_n}^2$ is a hamiltonian graph, so according to the theorem 1.4, $b_2(P_n^2) \leq [\frac{n}{3}]+2 = k+2$. Suppose, for the sake of contradiction, that $\{x_1, x_2, \ldots, x_b\}$ is a $2-$ burning sequence for $P_n^2$, which is $b \leq k+1$. Since for every $1 \leq i \leq b-1$, by selecting every vertex $x_i$ in each arbitrary $2-$ burnig sequence in step $i+1$, at most 2 vertices from $P_n^2$ to previously burned vertices is added, so the maximum number of burned vertices in an arbitrary optimal $2-$ burning sequence is equal to $5 + 3(b-3)$. Since $b \leq k+1$, at most $5 + 3(b-3) \leq 5 + 3(k+1-3) = 3k-1$ vertices are burned while $\mid V(G) \mid = 3k+1$, therefore it is a contradiction. Thus $b_2({P_n}^2) = k+2$.

□

**Theorem 1.7.** *If $n = 3k + i$ and $0 \leq i \leq 2$, then we have $b_2({C_n}^2) = k + 2$.*

*Proof.* Simillary to perviuos theorem.

□

**Theorem 1.8.** *i) If $G$= srg$(n, k, \lambda, \mu)$, then we have $b_2(G^2) = 3$.*
*ii) If $G = W_n$ and $n \geq 3$, then we have $b_2(G^2) = 4$.*

*Proof.* i) Since the square of these graphs is a complete graph, so according to the theorem 1.2 , $b_2(G^2) = 3$.

ii) The graph $G^2$ contains a subgraph ${W_{n-1}}^2$ whose vertices are a $2-$ dominating set for $G^2$. According to lemma 1.3, by first burning ${W_{n-1}}^2$ and then selectinging an unburned vertex, $G^2$ burns. Therefore, according to the previous theorem, we have

$$b_2(G^2) \leq b_2({W_{n-1}}^2) + 1 = 3 + 1 = 4.$$

On the other hand, it is easy to see that by selectinging every arbitrary $2-$ burning sequence length 3 , $G^2$ is not burned. So we have $b_2(G^2) = 4$. □

.

## References

[1] A. Bonato, J. Janssen and E. Roshanbin, How to burn a graph, Internet Mathematics 1-2 (2016) 85-100.

[2] A. Bonato, J. Janssen and E. Roshanbin, Burning a Graph as a Model of Social Contagion, Lecture Notes in Computer Science 8882 (2014) 13-22.

[3] H. Fleischner, The square of every two-connected graph is hamiltonian, Department of Mathematics, J.comb. TH.(B), 16(1974)323-339

[4] D. Mitsche, P. Pra l at and E. Roshanbin. Burning number of graph products. Theoret. Comput. Sci, 746(2018)124-135.

[5] Yinkui Li, Xiaoxiao Qin and Wen Li, The generalized burning number of graphs, Applied Mathematics and Computation 411 (2021) 126306.

[6] D.B. West, Introduction to Graph Theory, 2nd edition, Prentice Hall, (2001).

e-mail: abolbahmani@yahoo.com
e-mail: emami@znu.ac.ir
e-mail: o.naserian@gmail.com