On the Normality of t-Cayley Hypergraphs with Valency 3

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Abstract

A t-Cayley hypergraph $X = t\text{-}Cay(G,S)$ is called *normal* for a finite group G, if the right regular representation $R(G)$ of G is normal in the full automorphism group $Aut(X)$ of X. In this paper, we classify all normality of t-Cayley hypergraph, where G is a finite abelian group and $|S| = 3$.

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1 Introduction

A *hypergraph* X is a pair (V, E), where V is a finite nonempty set and E is a finite family of nonempty subsets of V. The elements of V are called *hypervertices* or simply *vertices* and the elements of E are called *hyperedges* or simply *edges*. Two vertices u and v are *adjacent* in hypergraph $X=(V,E)$ if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$ holds $e \cap f \neq 0$, we say that e and f are *adjacent*. A vertex v and an edge e are *incident* if $v \in e$. We denote by $X(v)$ the *neighborhood* of a vertex v, i.e. $X(v) = \{u \in V : \{u, v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or *d-regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph $X=(V,E)$ is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. A hypergraph is known as *uniform* or *k-uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertex is a 2-uniform hypergraph.

Let $X_1=(V_1,E_1)$ and $X_2=(V_2,E_2)$ be two hypergraphs. A *homomorphism* $\varphi : X_1 \to X_2$ is a map $\varphi : V_1 \to V_2$ that preserves adjacencies, that is, $\varphi(e) \in E_2$ for each $e \in E_1$. When φ is a bijection and its inverse map is also a homomorphism then φ is an *isomorphism* between the two hypergraphs and X_1 and X_2 are isomorphic.

An isomorphism from a hypergraph X onto itself is an *automorphism*. The *automorphism group* of X is denoted by $Aut(X)$.

For a group G and a subset S of G such that $1_G \notin S$ and $S = S^{-1} := \{s^{-1} | s \in S\}$, the *Cayley graph* $X = Cay(G,S)$ of G with respect to S is defined as the graph with vertex set $V(X) = G$, and edge set $E(X) = \{\{g,h\} | hg^{-1} \in S\}$.

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Obviously, the Cayley graph $Cay(G,S)$ has valency $|S|$, and it easily follows that $Cay(G,S)$ is connected if and only if $G=\langle S\rangle$, that is, S generates G. For a group G, denote $R(G)$ as the right regular representation of G. Define $Aut(G,S):=\{\alpha\in Aut(G)|S^{\alpha}=S\}$, acting naturally on G. Then, it is easy to see that each Cayley graph $X=Cay(G,S)$ admits the group $R(G).Aut(G,S)$ as a subgroup of automorphisms. Moreover (see [4]), $N_{Aut(X)}(R(G))=R(G).Aut(G,S)$. Note that $R(G)\cong G$. So we can identify G with $R(G)\le Aut(X)$ for $X=Cay(G,S)$. The Cayley graph $X=Cay(G,S)$ is called *normal* if G is normal in $Aut(X)$. In this case $Aut(X)=G.Aut(G,S)$.

Let G be a group and let S be a set of subsets $s_1,s_2,...,s_n$ of $G-\{1_G\}$ such that $G=\langle\bigcup_{i=1}^{n}s_i\rangle$, that is, $\bigcup_{i=1}^{n}s_i$ generates G. A *Cayley hypergraph* $CH(G,S)$ has vertex set G and edge set $\{\{g,gs\}|g\in G,s\in S\}$, where an edge $\{g,gs\}$ is the set $\{g\}\cup\{gx|x\in s\}$. For all $s\in S$, if $|s|=1$, then the Cayley hypergraph is a Cayley graph. Therefore a Cayley hypergraph is a generalization of a Cayley graph [5]. Also, Lee and Kwon [5] proved that a hypergraph X is Cayley if and only if $Aut(X)$ contains a subgroup which acts regularly on the vertex set of X.

In 1994, Buratti [3] introduced the concept of a t-Cayley hypergraph as follows. Let G be a finite group, S a subset of $G-\{1_G\}$ and t an integer satisfying $2\le t\le$max$\{o(s)|s\in S\}$. The *t-Cayley hypergraph* $X=t\text{-}Cay(G,S)$ of G with respect to S is defined as the hypergraph with vertex set $V(X)=G$, and for $E\subseteq G$,

$$E\in E(X)\Longleftrightarrow\exists g\in G,\exists s\in S:E(X)=\{gs^{i}|0\le i\le t-1\}.$$

Note that any 2-Cayley hypergraph is a Cayley graph and vice versa. For any $s_i\in S$, if $s_i=\{s,...,s^{t-1}\}$ for some $s\in G-\{1_G\}$, then the Cayley hypergraph $CH(G,S)$ is a t-Cayley hypergraph $t-Cay(G,S)$. Hence a Cayley hypergraph is a generalization of a t-Cayley hypergraph. In fact every t-Cayley hypergraph is a subclass of the more general Cayley hypergraphs, or *group hypergraphs* which is defined by Shee in [6].

The concept of normality of the Cayley graph is known to be of fundamental importance for the study of arc transitive graphs. So, for a given finite group G, a natural problem is to determine all the normal or non-normal Cayley graph of G. Some meaningful results in this direction, especially for the undirected Cayley graphs, have been obtained. Baik et al. [1] determined all non-normal Cayley graphs of abelian groups of valency at most 4 and later [2] dealt with valency 5. For directed Cayley graphs, Xu et al. [7] determined all non-normal Cayley graphs of abelian groups of valency at most 3. In this paper, we classify all normality of t-Cayley hypergraph, where G is a finite abelian group and $|S|=3$.

2 Main results

Proposition 2.1. *Let G be a finite group, and let S be a generating set of G not containing the identity 1_G, and α an automorphism of G. Then t-Cayley hypergraph $X=t\text{-}Cay(G,S)$ is normal if and only if $X'=t\text{-}Cay(G,S^{\alpha})$ is normal.*

Proof. Let $A'=Aut(X')$. It will be shown that (1) $\alpha^{-1}A\alpha=A'$, and (2) $\alpha^{-1}R(G)\alpha=R(G)$. For the first equation, we suppose that $\alpha^{-1}\rho\alpha\in\alpha^{-1}A\alpha$, where $\rho\in A$. Now if $E'\in E(X')$, then $E'=\{xs^{i}|0\le i\le t-1\}$ for some $x\in G$ and $s\in S$. Therefore

$$\begin{aligned}(E')^{\alpha^{-1}\rho\alpha} &= \{(xs^{i})^{\alpha^{-1}\rho\alpha}|0\le i\le t-1\}\\ &= \{x^{\alpha^{-1}},x^{\alpha^{-1}}(s)^{\alpha^{-1}},...,x^{\alpha^{-1}}(s^{t-1})^{\alpha^{-1}}\}^{\rho\alpha}.\end{aligned}$$

It follows that,

$$(E^{'})^{\alpha^{-1}\rho\alpha} = \{y, ys', y(s^{'})^2, ..., y(s^{'})^{t-1}\}^{\rho\alpha},$$

where $s^{'} = s^{\alpha^{-1}}$ and $x^{\alpha^{-1}} = y$. Since $\rho \in A$,

$$(E^{'})^{\alpha^{-1}\rho\alpha} = \{z, zs^{''}, ..., z(s^{''})^{t-1}\}^{\rho} \in E(X^{'}),$$

where $s^{''} = (s^{'})^{\alpha}$ and $y^{\alpha} = z$. With the similar argument $A^{'} \subseteq \alpha^{-1}A\alpha$ and so $\alpha A \alpha^{-1} = A^{'}$. Also it is easy to see that $\alpha^{-1}R(G)\alpha = R(G)$. Now X is normal, that is, $R(G) \lhd A$ if and only if $R(G) = \alpha^{-1}R(G)\alpha \lhd \alpha^{-1}A\alpha = A'$. □

By considering the above proposition, the following result is obtained.

Proposition 2.2. *Let G be a finite abelian group, and let S be a generating set of G not containing the identity 1_G. Assume S satisfies the condition s, t, u, $v \in S$ with*

$$st = uv \neq 1 \Rightarrow \{s, t\} = \{u, v\}. \tag{1}$$

Then the t-Cayley hypergraph is normal.

We omit the easy proof of the following lemma.

Lemma 2.3. *Let $G = G_1 \times G_2$ be the direct product of two finite groups G_1 and G_2, S_1 and S_2 subsets of G_1 and G_2, respectively, and $S = S_1 \cup S_2$ the disjoint union of S_1 and S_2. Let $t, t^{'}, t^{''}$ be integers where t=max$\{t^{'}, t^{''}\}$. Then*

(i) t-Cay$(G, S) \cong t^{'}$-Cay$(G_1, S_1) \times t^{''}$-Cay(G_2, S_2).

(ii) If t-Cay(G, S) is normal, then $t^{'}$-Cay(G_1, S_1) is also normal.

(iii) If $t^{'}$-Cay(G_1, S_1) and $t^{''}$-Cay(G_2, S_2) are both normal and relatively prime, then t-Cay(G, S) is normal.

From Lemma 2.3, we have the following.

Lemma 2.4. *If $T \cap \langle J \rangle = 1$ and J is independent, then $G = T \times \mathbb{Z}_2^J$ and $X = Y \times$ t-Cay$(\langle J \rangle, J)$. Moreover, if Y is normal and relatively prime with K_2, then X is normal.*

Now the conditions are ready to give a proof for the following theorem where is the main result of this paper.

Theorem 2.5. *Let $X = t$-Cay(G, S) be a connected t- Cayley hypergraph of an abelian group G on S with the valency 3. Then X is normal except one of the following cases happens:*

1. *$X = 2n$-Cay$(\mathbb{Z}_{2n} \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle, \{a, a^{n+1}, b\})$, where $n > 2$, $m > 1$.*

2. *$X = n$-Cay$(\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \{a, ab, c\})$, where $n > 2$, $m > 1$.*

3. $X = 2n\text{-}Cay(\mathbb{Z}_{2n} = \langle a \rangle, \{a, a^{n+1}, a^n\})$, *where* $n > 2$.

4. $X = n\text{-}Cay(\mathbb{Z}_n \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, ab, b\})$, *where* $n > 2$.

5. $X = 2k\text{-}Cay(\mathbb{Z}_{2k} \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, ab, a^k\})$, *where* $k > 2$.

6. $X = 2k\text{-}Cay(\mathbb{Z}_{2k} \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, ab, a^k b\})$, *where* $k > 2$.

7. $X = 4n\text{-}Cay(\mathbb{Z}_{4n} = \langle a \rangle, \{a, a^{2n+1}, a^{n+1}\})$, *where* $n = 4k+1, k > 0$.

8. $X = 4n\text{-}Cay(\mathbb{Z}_{4n} \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, a^{2n+1}, a^{n+1}b\})$, *where* $n = 2k+1, k > 0$.

9. $X = n\text{-}Cay(\mathbb{Z}_n \times \mathbb{Z}_4 = \langle a \rangle \times \langle b \rangle, \{a, ab^2, ab\})$, *where* $n = 4k, k > 0$.

10. $X = k\text{-}Cay(\mathbb{Z}_k \times \mathbb{Z}_t = \langle a \rangle \times \langle b \rangle, \{a^{k/nh}b, a^{k/nh}bc, a^{k/mh}b^{-1}\})$, *where* $c = (a^{k/nh}b)^{nh/2}$.

11. $X = k\text{-}Cay(\mathbb{Z}_k \times \mathbb{Z}_t \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \{a^{k/nh}b, a^{k/nh}bc, a^{k/mh}b^{-1}\})$.

In cases (10) and (11), $k = \frac{mnh}{(m,n)}$ and $t = (m, n)$.

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