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On a Class of Symplectic Graphs and Their Automorphisms

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Abstract

It easy to see that each graph is a modification of a reduced graph Γ of the same rank. It is proved that for every reduced graph with binary rank $2r$, there is a unique maximal graph with binary rank $2r$ which conatins Γ as an induced subgraph. These maximal graphs are called symplectic graphs. In this paper, we study the symplectic graphs which are defined over a ring. We also find the automorphism group of symplectic graphs which are defined over $\mathbb{Z}_{p^n}$, where p is a prime number and n is positive integer.

Keywords: Automorphism; Symplectic Graph; Symplectic Group; Generalized Symplectic Graph.

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1 Introduction

In this paper, a graph $\Gamma = \Gamma(\mathcal{V}, E)$ is considered as a simple undirected graph with vertex-set $V(\Gamma) = \mathcal{V}$, and edge-set $E(\Gamma) = E$.

In this paper, let R be a commutative ring with identity element 1, and let V be a free R - module of R - dimension $n \geq 2$. The symplectic form β is a bilinear form $\beta : V \times V \longrightarrow R$, such that $\beta(x, x) = 0$ for all $x \in V$. The pair (V, β) is called a symplectic space. The symplectic form $\beta : V \times V \longrightarrow R$ is called nonsingular, when the R-module homomorphism from V to $V^* = Hom_R(V, R)$ given by $x \longmapsto \beta(\ , x)$ is an isomorphism, for all $x \in V$. In the sequence, assume that β is a nonsingular symplectic form.

Recall that an element x in V is unimodular if there is an $f \in V^*$ such that $f(x) = 1$. For $x \in V$, we call Rx a line. A hyperbolic pair$\{x, y\}$ is a pair of unimodular vectors in V with the property that $\beta(x, y) = 1$. The module $H = Rx \bigoplus Ry$ is called a hyperbolic plane.

Any unimodular vector $u \in V$ may be complemented to a hyperbolic pair as follow:
Since u is unimodular, there is an $f \in V^*$ with $f(u) = 1$. Since β is nonsingular, there is an v in V with $1 = f(u) = \beta(u, v)$. Then, $\{u, v\}$ is a hyperbolic pair. A ring R is stably free whenever $V = V_1 \oplus P$, V and V_1 are free R - modules, then P is a free R - module.

[1]speaker

Proposition 1.1. *[?] Suppose that R is a stably free ring, and V be a symplectic space over R. Then V is an orthogonal direct sum $V = H_1 \perp H_2 \perp \ldots \perp H_m$ of hyperbolic planes $H_1, H_2, \ldots, H_m$. In particular, the dimension of V is even.*

Lemma 1.2. *[?] Let x and y be unimodular elements in V. Then $Rx = Ry$ if and only if $x = \lambda y$ for some $\lambda \in R^*$.*

Let $\Gamma(\mathcal{V}, E)$ and $\Lambda(\mathcal{V}', E')$ be two graphs. The mapping $\alpha : \mathcal{V} \longrightarrow \mathcal{V}'$ is a homomorphism from Γ to Λ if $v, w \in V(\Gamma)$ are adjacent in Γ, then $\alpha(v), \alpha(w) \in V'(\Lambda)$ are adjacent in Λ. An isomorphism between Γ and Λ is a bijection homomorphism $\alpha : \mathcal{V} \longleftrightarrow \mathcal{V}'$ with $v, w \in V(\Gamma)$ are adjacent in Γ, if and only if $\alpha(v), \alpha(w) \in \mathcal{V}'(\Lambda)$ are adjacent in Λ.
An automorphism of a graph Γ is an isomorphism from Γ to itself. The set of all automorphisms of Γ, with composition of functions, is called the automorphism group of Γ and denoted by $Aut(\Gamma)$.

In most situations, it is difficult to determine the automorphism group of a graph, but there are various in the literature and some of the recent works come in the references [?, ?]. Now, let Γ be a graph with automorphism group $G = Aut(\Gamma)$. For vertex $v \in V(\Gamma)$, let G_v denote the stabilizer subgroup of vertex v; that is, the subgroup of G containing of those automorphism that fix v. From first isomorphism theorem, we know that:

$$[G : G_v] = \frac{|G|}{|G_v|} \leq |V(\Gamma)|.$$

2 symplectic and generalized symplectic group

Suppose that (V, β) and (V', β') are two symplectic spaces. An isometry from (V, β) to (V', β') is an R - isomorphism $\sigma : V \longrightarrow V'$ such that:

$$\beta(x_1, x_2) = \beta'(\sigma(x_1), \sigma(x_2)) \text{ for every elements } x_1, x_2 \in V.$$

It is easy to verify that the set of all isometries from (V, β) to (V, β) is a group; this group is called symplectic group over V and denoted by $SP_R(V)$.

Definition 2.1. Let $B = \{v_1, \ldots, v_{2n}\}$ be a basis for the symplectic space (V, β). The matrix $B = (b_{ij})_{1 \leq i,j \leq 2n}$, where $b_{ij} = \beta(v_i, v_j)$ is called the matrix of the form β over B.

The following theorem has been obtained from the definition of symplectic space and has an easy proof.

Theorem 2.2. *Let (V, β) and (W, β') be two symplectic spaces with $dimV = dimW = 2n$. Suppose that B_1 and B_2 are ordered basis of V and W respectively. If we denote the matrices of β and β' with respects to the above basis by B and C respectively, then $T : V \longrightarrow W$ is an isometry from V to W if and only if $A^tCA = B$, where A is matrix of T with respect to B_1.*

Corollary 2.3. *Let R be a stably free ring and (V, β) be symplectic space over R. Then*

$$SP_R(V) = \{A | A \text{ is invertable and } A^tJA = J\}$$

where J is blockdiagonal matrix as follow:

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

In [?] it is proved that, $Z(SP_R(V)) = \{\pm I_{2n}\}$, where $Z(SP_R(V))$ denotes the center of the group $SP_R(V)$. A commutative ring R have a stable range one if for all $\alpha, \beta \in R$ with $\langle \alpha, \beta \rangle = R$, there exist a δ in R such that $\alpha + \delta\beta \in R^*$.

Lemma 2.4. *[?] Let R be a commutative ring with stable range 1 and $2 \in R$ be an unit. Let V be a symplectic space over R. Then $SP_R(V)$ acts transitively on unimodular vectors and on hyperbolic planes.*

Definition 2.5. Generalized symplectic group over ring R is denoted by $GSP_R(V)$ and defined as follow:

$$GSP_R(V) = \{T | T \text{ is invertible over } R \text{ and } TJT^t = kJ \text{ for some } k \in R^*\}.$$

3 symplectic graphs

For all terminologies and notations not defined here, we follow [?, ?]. We now define a class of regular graphs, which is known as symplectic graphs.

Definition 3.1. Let (V, β) be a symplectic space over ring R. The symplectic graph over $SP_R(V)$ denoted by $\mathcal{G}SP_R(V)$, is a graph with vertex- set

$$\{Rx | x \text{ is unimodular in } V\},$$

and two vertices Rx and Ry are adjacent if and only if $\beta(x, y) \in R^*$.

This adjacency is well defined, since if x_1, x_2, y_1, y_2 are unimodular elements in V with $Rx_1 = Rx_2$ and $Ry_1 = Ry_2$, then there exist $\lambda, \mu \in R^*$ such that $x_1 = \lambda x_2$ and $y_1 = \mu y_2$. Therefore

$$\beta(x_1, y_1) \in R^* \Longleftrightarrow \beta(\lambda x_2, \mu y_2) \in R^*$$
$$\Longleftrightarrow \lambda\mu\beta(x_2, y_2) \in R^* \Longleftrightarrow \beta(x_2, y_2) \in R^*.$$

Now from lemma **??** we have the following lemma that proved in [?].

Lemma 3.2. *Let R be a commutative ring with stable range 1 and $2 \in R$ be an unit. Then the symplectic graph $\mathcal{G}SP_R(V)$ is vertex-transitive and edge-transitive.*

We now define a symplectic graph over $R = \mathbb{Z}_{p^n}$. Let $V^{2v} \subseteq \mathbb{Z}_{p^n}^{(2v)}$ be a set of elements $(a_1, a_2, \ldots, a_{2v})$, where for all $1 \leq i \leq 2v$, $a_i \in \mathbb{Z}_{p^n}$ and there is an $i \in \{1, \ldots, 2v\}$ such that a_i is invertible in $\mathbb{Z}_{p^n}$. We define an equivalence relation $\sim_{p^n}$ on V by the following rule:

$$(a_1, a_2, \ldots, a_{2v}) \sim_{p^n} (b_1, b_2, \ldots, b_{2v}) \Longleftrightarrow (a_1, a_2, \ldots, a_{2v}) = \lambda(b_1, b_2, \ldots, b_{2v}),$$

for some $\lambda \in \mathbb{Z}_{p^n}^*$.
Let $[a_1, \ldots, a_{2v}]$ denotes the equivalence class of $(a_1, \ldots, a_{2v})$ with respect to $\sim_{p^n}$, and let $V_{\sim_{p^n}}^{(2v)}$ be the set of all equivalence classes. We define the bilinear form $\beta : V_{\sim_{p^n}}^{(2v)} \times V_{\sim_{p^n}}^{(2v)} \longrightarrow R$ by the rule $\beta(x, y) = xJy^t$.
The symplectic graph module p^n on $\mathbb{Z}_{Pn}^{(2v)}$, relative to J which is denoted by $SP_{p^n}^{(2v)}$, is a graph with vertex-set $\{[a_1, \ldots, a_{2v}] | (a_1, \ldots, a_{2v}) \in V^{(2v)}\}$ and adjacency defined by

$$[a_1, \ldots, a_{2v}] \text{ adjacent to } [b_1, \ldots, b_{2v}] \text{ if and only if } \beta(x, y) \in \mathbb{Z}^*_{P^n},$$

where $x = (a_1, \ldots, a_{2v})$ and $y = (b_1, \ldots, b_{2v})$. In [?], it is proves that $SP^{(2v)}_{p^n}$ is a vertex and edge-transitive graph.
In the first step, note that β is a symplectic form over $\mathbb{Z}^{(2v)}_{P^n}$.

Lemma 3.3. *Each element of* $V := V^{(2v)}_{\sim p^n}$ *is unimodular.*

Proof. If we define $q : V \longrightarrow V^*$ by $q(x) = q_x$ where $q_x(v) = \beta(x, v)$, then q is an isomorphism. For $x = (a_1, \ldots, a_{2v})$, let a_i be invertible in $\mathbb{Z}_{P^n}$. If $i \geq v + 1$, then let $y = (0, \ldots, b_{i-v} = 1, 0, \ldots, 0)$ and so $\beta(x, y) = a_i b_{i-v} = 1$. If $i \leq v$, then let $y = (0, \ldots, b_{i+v} = 1, 0, \ldots, 0)$ and so $\beta(x, y) = a_i b_{i+v} = 1$. Then there is an $f = q_y \in V^*$ such that $q_y(x) = f(x) = 1$ and hence x is unimodular. □

By previous lemma, we conclude that for $R = \mathbb{Z}_{P^n}$, $\mathcal{G}SP_R(v)$ is isomorphic to $SP^{(2v)}_{P^n}$.
In [?], it is proved that $\mathbb{Z}_{P^n}$ has a stable range one, and we know that for $p \geq 2$, 2 is unit in $\mathbb{Z}_{P^n}$, where p is prime. Then by lemma **??**. we conclude that $SP^{(2v)}_{P^n}$ is vertex-transitive and edge-transitive.

Lemma 3.4. *Let p be a prime integer and $R = \mathbb{Z}_{P^n}$ and $V = \mathbb{Z}^{(2v)}_{P^n}$. Suppose that $T \in GSP_R(V)$. We define $\sigma_T : V \longrightarrow V$ by the rule $\sigma_T(x) = R(xT)$ for all unimodular elements $x \in V$. Then $T \in GSP_R(V)$ if and only if $\sigma_T \in Aut(\mathcal{G}SP_R(V))$.*

Proof. Let $T \in GSP_R(V)$ and $R\alpha, R\beta \in SP_R(V)$, then for $T \in GSP_R(V)$ we have $TJT^t = kJ$, where $k \in \mathbb{Z}^*_{P^n}$. Then $\alpha J \beta^t = k^{-1} \alpha TJT^t \beta^t$ and $R\alpha$ is adjacent to $R\beta$ if and only if αT is adjancent to βT, hence $\sigma_T \in Aut(\mathcal{G}SP_R(V))$.
Conversely, assume that $\sigma_T \in Aut(\mathcal{G}SP_R(V))$, then

$$R\alpha \nsim R\beta \Longleftrightarrow \alpha J \beta^t \nsubseteq R^* \Longleftrightarrow \alpha J \beta^t = r,$$

for some $r \in \mathbb{Z}_{P^n} \setminus \mathbb{Z}^*_{P^n}$.
If $r = 0$, then $\alpha J \beta^t = 0$ if and only if $\alpha(TJT^t)\beta^t = 0$. Hence, for any nonzero $\alpha \in R$, two equations $(\alpha J)X^t = 0$ and $(\alpha TJT^t)X = 0$ have the same solutions. But $rank(\alpha J) = rank(\alpha TJT^t) = 1$, and so $\alpha k = s\alpha(TJT^t)$ for some $s \in R^*$.
Now let $\{e_1, \ldots, e_{2v}\}$ be the standard basis for V, then we obtine

$$J = diag(k_1, \ldots, k_{2v})TJT^t,$$

for some $k_1, \ldots, k_{2v} \in R^x$. If we put $\alpha = (1, \ldots, 1)$, then $k_1 = k_2 = \ldots = k_{2v} = k \in R^x$, and so $J = KTJT^t$, $T \in GSP_R(V)$.
If $\alpha J \beta^t = r \neq 0$, then $r = P^n$ for $1 \leq m \leq n$, and $P^{n-m} \alpha J \beta^t = P^n = 0$, so we can do as above and then $T \in GSP_R(V)$.

□

We now proceed to proving the main result of this paper.

Theorem 3.5. *Let $R = \mathbb{Z}_{P^n}$ and $V = \mathbb{Z}^{(2v)}_{P^n}$, then*

$$Aut(\mathcal{G}SP_R(V)) = \frac{GSP_R(V)}{kI},$$

for some $k \in R^*$.

Proof. We define the homomorphism $\sigma : GSP_R(V) \longrightarrow Aut(\mathcal{G}SP_R(V))$ by $T \longmapsto \sigma_T$. In [**?**], it is proved that, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = kT_2$, for $k \in R^*$. Then $\ker\sigma = \{kI |\, k \in R^*\}$. Now it remains to show that for any $f \in Aut(\mathcal{G}SP_R(V))$, there is an $T \in GSP_R(V)$, such that $f = \sigma_T$. For any $\alpha \neq 0$ in V, we will denote $f(R\alpha \setminus \{0\})$ by $f(\alpha)$ and $f(0) = 0$. Since $f \in Aut(\mathcal{G}SP_R(V))$, then $\alpha J\beta^t = f(\alpha)J(f(\beta))^t$ for any $\alpha, \beta \in V$. Fix $\alpha \in V$ and let $\beta_1, \beta_2 \in V$, then $\alpha J\beta_1^t = f(\alpha)J(f(\beta_1))^t$ and $\alpha J\beta_2^t = f(\alpha)J(f(\beta_2))^t$ then $\alpha J(\beta_1 + \beta_2)^t = f(\alpha)J(f(\beta_1)+f(\beta_2))^t$. Thus, $\alpha J(\beta_1+\beta_2)^t = f(\alpha)J(f(\beta_1+\beta_2))^t$, hence $f(\alpha)J(f(\beta_1+\beta_2)-f(\beta_1)-f(\beta_2)) = 0$ and therefor for all $\alpha \in V$, we have $f(\beta_1 + \beta_2) = f(\beta_1) + f(\beta_2)$. Let,

$$T = \begin{pmatrix} f(1,0,\dots,0) \\ f(0,1,\dots,0) \\ \vdots \\ f(0,0,\dots,1) \end{pmatrix}$$

Therefor $f(\alpha) = \alpha T$, for any $\alpha \in V$. Then T is nonsingular, so by lemma **??**. $T \in SP_R(V)$ and $f = \sigma_T$.

□

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