8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

Integral inequalities for differentiable (h, m)-convex functions with generalized Caputo-type derivatives

Paulo M. Guzmán[1]

Abstract

In this work we obtain integral inequalities of the Hermite-Hadamard type, using generalized derivatives of the Caputo type. Throughout the work, we see that several results reported in the literature are particular cases of those presented here.

Keywords: Integral inequalities, (h, m)-convex functions, Generalized Caputo type derivatives

AMS Mathematical Subject Classification [2010]: 26A33, 26D10, 47A63

1 Introduction

In Mathematics, the notion of convex function plays a very prominent role, due to its multiple applications and its theoretical overlaps with various other areas of science.

One of the most important inequalities for convex functions is the well-known Hermite-Hadamard inequality:

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\psi(x)dx \leq \frac{\psi(\nu_1)+\psi(\nu_2)}{2}.$$

This inequality holds for any convex ψ function on the interval $[\nu_1, \nu_2]$. Gives an estimate of the mean value of a convex function.

In [2] the following definitions were presented:

Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If the inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi\left(\frac{\varsigma}{m}\right)$$

holds for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$, then the function ψ is called (h, m)-modified convex of the first type in I.

Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If the inequality holds

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h(\tau))^s\psi\left(\frac{\varsigma}{m}\right)$$

[1]speaker

for all $\xi, \varsigma \in I$ and $\tau \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$, then the function ψ is called (h,m)-modified convex of the second type in I.

The differential operators that we will use in our work are the following:

Definition 1.1. Let $\alpha > 0$, and $\alpha \neq 1,2,3,,$ $n = [\alpha]+1$, $f \in AC^n[a,b]$, the space of functions that have the n-th absolutely continuous derivatives. The weight Caputo derivatives of the right-hand side and the left-hand side of order α are defined as follows:

$$\left({}_n^C D_{v_1+}^{w'} f\right)(v_2) = \int_{v_1}^{v_2} w' \left[\frac{v_2 - x}{\frac{v_2-v_1}{r+1}}\right] f^{(n)}(x)dx,$$

$$\left({}_n^C D_{v_2-}^{w'} f\right)(v_1) = \int_{v_1}^{v_2} w' \left[\frac{x - v_1}{\frac{v_2-v_1}{r+1}}\right] f^{(n)}(x)dx.$$

In this work we obtain different variants of the Hermite-Hadamard inequality, within the framework of modified (h,m)-convex functions, using generalized operators.

2 Main results

Theorem 2.1. *Let f be a positive function such that $f \in C^n[v_1,v_2]$. If $f^{(n)}$ is a modified (h,m)-convex function of the second type with $m \in (0,1]$ and $0 < v_1 < mv_2 < +\infty$, then we have the following inequality:*

$$\begin{aligned} & f^{(n)}\left(\frac{v_1+v_2}{2}\right)\int_0^1 w(t)dt \qquad (1)\\ \leq\ & h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(v_2-v_1)}\left({}_n^C D^w_{\left(\frac{v_1+rv_2}{r+1}\right)+} f\right)(v_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s \frac{(r+1)}{(v_2-v_1)}\left({}_n^C D^w_{\left(\frac{rv_1+v_2}{r+1}\right)-} f\right)(v_1)\\ \leq\ & \left[h^s\left(\frac{1}{2}\right)f^{(n)}(v_1) + \left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}(v_2)\right]\int_0^1 w(t)h^s\left(\frac{t}{r+1}\right)dt\\ +\ & m\left[h^s\left(\frac{1}{2}\right)f^{(n)}\left(\frac{v_2}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}\left(\frac{v_1}{m}\right)\right]\int_0^1 w(t)\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \end{aligned}$$

Proof. For $x, y \in [0,+\infty)$, $t = \frac{1}{2}$ and $m = 1$, we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \leq h^s\left(\frac{1}{2}\right)f^{(n)}(x) + \left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}(y).$$

Making $x = \frac{t}{r+1}v_1 + \left(\frac{r+1-t}{r+1}\right)v_2$,$y = \frac{t}{r+1}v_2 + \left(\frac{r+1-t}{r+1}\right)v_1$, with $t \in [0,1]$, we have

$$\begin{aligned} f^{(n)}\left(\frac{v_1+v_2}{2}\right) \leq\ & h^s\left(\frac{1}{2}\right)f^{(n)}\left(\frac{t}{r+1}v_1 + \left(\frac{r+1-t}{r+1}\right)v_2\right) \qquad (2)\\ +\ & \left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}\left(\frac{t}{r+1}v_2 + \left(\frac{r+1-t}{r+1}\right)v_1\right). \end{aligned}$$

Multiplying both members of the previous inequality by $w(t)$, integrating with respect to the variable t between 0 and 1, and changing the variables, we obtain the first inequality of (1).

$$
\begin{aligned}
f^{(n)}\left(\frac{v_1+v_2}{2}\right)\int_0^1 w(t)dt &\leq h^s\left(\frac{1}{2}\right)\int_0^1 w(t)f^{(n)}\left(\frac{t}{r+1}v_1+\left(\frac{r+1-t}{r+1}\right)v_2\right)dt \\
&+ \left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 w(t)f^{(n)}\left(\frac{t}{r+1}v_2+\left(\frac{r+1-t}{r+1}\right)v_1\right)dt, \\
f^{(n)}\left(\frac{v_1+v_2}{2}\right)\int_0^1 w(t)dt &\leq h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(v_1-v_2)}\int_{v_2}^{\frac{v_1+rv_2}{r+1}} w\left[\frac{(x-v_2)}{\frac{v_1-v_2}{(r+1)}}\right]f^{(n)}(x)\,dx \\
&+ \left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(v_2-v_1)}\int_{v_1}^{\frac{rv_1+v_2}{r+1}} w\left[\frac{(x-v_1)}{\frac{(v_2-v_1)}{(r+1)}}\right]f^{(n)}(x)\,dx, \\
f^{(n)}\left(\frac{v_1+v_2}{2}\right)\int_0^1 w(t)dt &\leq h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(v_2-v_1)}\int_{\frac{v_1+rv_2}{r+1}}^{v_2} w\left[\frac{(v_2-x)}{\frac{(v_2-v_1)}{(r+1)}}\right]f^{(n)}(x)\,dx \\
&+ \left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(v_2-v_1)}\int_{v_1}^{\frac{rv_1+v_2}{r+1}} w\left[\frac{(x-v_1)}{\frac{(v_2-v_1)}{(r+1)}}\right]f^{(n)}(x)\,dx, \\
f^{(n)}\left(\frac{v_1+v_2}{2}\right)\int_0^1 w(t)dt &\leq h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(v_2-v_1)}\left({}^C_nD^w_{\left(\frac{v_1+rv_2}{r+1}\right)+}f\right)(v_2) \\
&+ \left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(v_2-v_1)}\left({}^C_nD^w_{\left(\frac{rv_1+v_2}{r+1}\right)-}f\right)(v_1).
\end{aligned}
$$

From the right side of (2), we obtain

$$
\begin{aligned}
&h^s\left(\frac{1}{2}\right)f^{(n)}\left(\frac{t}{r+1}v_1+\left(\frac{r+1-t}{r+1}\right)v_2\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}\left(\frac{t}{r+1}v_2+\left(\frac{r+1-t}{r+1}\right)v_1\right)= \\
&h^s\left(\frac{1}{2}\right)f^{(n)}\left(\frac{t}{r+1}v_1+m\left(\frac{r+1-t}{r+1}\right)\frac{v_2}{m}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s f^{(n)}\left(\frac{t}{r+1}v_2+m\left(\frac{r+1-t}{r+1}\right)\frac{v_1}{m}\right)\leq \\
&h^s\left(\frac{1}{2}\right)\left[f^{(n)}(v_1)\,h^s\left(\frac{t}{r+1}\right)+mf^{(n)}\left(\frac{v_2}{m}\right)\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right]+ \\
&\left(1-h\left(\frac{1}{2}\right)\right)^s\left[f^{(n)}(v_2)\,h^s\left(\frac{t}{r+1}\right)+mf^{(n)}\left(\frac{v_1}{m}\right)\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right].
\end{aligned}
$$

Multiplying by $w(t)$, integrating with respect to the variable t between 0 and 1, we obtain the right side of (1). In this way, the proof is complete. □

The following result would be very useful to prove future theorems.

Lemma 2.2. *Let f be a real function defined on the real interval $[v_1, v_2]$ and differentiable on (v_1, v_2). If $f' \in L_1(v_1, v_2)$, and $w(t)$ is a function differentiable on (v_1, v_2), then we have the following equality:*

$$
\begin{aligned}
&\left\{-w(1)\left(f^{(n)}\left(\frac{v1+rv_2}{r+1}\right)+f^{(n)}\left(\frac{rv_1+v_2}{r+1}\right)\right)+w(0)\left(f^{(n)}(v_1)+f^{(n)}(v_2)\right)\right\} \qquad (3)\\
&+\frac{r+1}{v_2-v_1}\left[\left({}^C_nD^{w'}_{\left(\frac{rv_1+v_1}{r+1}\right)-}f\right)(v_1)+\left({}^C_nD^{w'}_{\left(\frac{v_1+rv_1}{r+1}\right)+}f\right)(v_2)\right] \\
&=\frac{v_2-v_1}{r+1}\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)-f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt.
\end{aligned}
$$

Proof. First of all let us note that,

$$\begin{aligned}
&\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)-f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt\\
=&\int_0^1 w(t)f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)dt-\int_0^1 f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)dt\\
=&\ I_1-I_2.
\end{aligned}$$

Integrating by parts, we have

$$I_1=\frac{r+1}{v_2-v_1}\left[-w(1)f^{(n)}(\frac{v_1+rv_2}{r+1})+w(0)f^{(n)}(v_2)\right]+\frac{(r+1)^2}{(v_2-v_1)^2}\int_{v_1}^{\frac{rv_1+v_2}{r+1}} w'\left[\frac{x-v_1}{\frac{v_2-v_1}{r+1}}\right]f^{(n)}(x)dx,$$

because

$$\int_0^1 w'(t)f^{(n)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)dt=\frac{n+1}{v_2-v_1}\int_{v_1}^{\frac{rv_1+v_2}{r+1}} w'\left[\frac{x-v_1}{\frac{v_2-v_1}{r+1}}\right]f^{(n)}(x)dx.$$

In an analogous way,

$$I_2=\frac{r+1}{v_2-v_1}\left[w(1)f^{(n)}(\frac{rv_1+v_2}{r+1})-w(0)f^{(n)}(v_1)\right]-\frac{(r+1)^2}{(v_2-v_1)^2}\int_{\frac{v_1+rv_2}{r+1}}^{v_2} w'\left[\frac{v_2-x}{\frac{v_2-v_1}{r+1}}\right]f^{(n)}(x)dx.$$

By making I_1-I_2, we obtain the desired equality. □

Using Lemma 2.2, we obtain the following results.

Theorem 2.3. *Let f be a positive real function defined on $[v_1,v_2]\subset\mathbb{R}$, such that $f^{(n)}\in L_1(v_1,mv_2)$. If $|f^{(n)}|$ es (h,m)-modified convex of the second type in $\left[v_1,\frac{v_2}{m}\right]$, we have the following inequality:*

$$\mathbb{I}\left(v_1,v_2,f,w\right)\ \le\ \frac{v_2-v_1}{r+1}\left(|f^{(n+1)}\left(v_1\right)|+|f^{(n+1)}\left(v_2\right)|\right)\mathbb{G}+m\left(|f^{(n+1)}\left(\frac{v_1}{m}\right)|+|f^{(n+1)}\left(\frac{v_2}{m}\right)|\right)\mathbb{H}. \tag{4}$$

where $\mathbb{I}(v_1,v_2,f,w)$ is the absolute value of the left side of (3),

$$\mathbb{G}=\int_0^1 |w(t)|h^s\left(\frac{t}{r+1}\right)dt$$

y

$$\mathbb{H}=\int_0^1 |w(t)|\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt.$$

Proof. From Lemma 2.2, we have

$$\begin{aligned}
&\left|\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)-f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt\right|\\
\le&\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)|dt+\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)|dt.
\end{aligned}$$

Using the modified (h,m)-convexity of $|f^{(n+1)}|$, we have

$$\begin{aligned}
&\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)|dt\\
\leq &\int_0^1 |w(t)|\left[h^s\left(\frac{t}{r+1}\right)|f^{(n+1)}(v_1)|+m\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s|f^{(n+1)}\left(\frac{v_2}{m}\right)|\right]dt\\
= &|f^{(n+1)}(v_1)|\int_0^1 |w(t)|h^s\left(\frac{t}{r+1}\right)dt+m|f^{(n+1)}\left(\frac{v_2}{m}\right)|\int_0^1 |w(t)|\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \qquad (5)
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)|dt\\
\leq &|f^{(n+1)}(v_2)|\int_0^1 |w(t)|h^s\left(\frac{t}{r+1}\right)dt+m|f^{(n+1)}\left(\frac{v_1}{m}\right)|\int_0^1 |w(t)|\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt.
\end{aligned}$$

From (5) and (6), we obtain (4). □

The above result can be improved if we impose additional conditions on $|f^{(n+1)}|^q$.

Theorem 2.4. *Let f be a positive real function defined on $[v_1,v_2]\subset\mathbb{R}$, such that $f^{(n+1)}\in L_1(v_1,mv_2)$. If $|f^{(n+1)}|^q$ is a modified convex function of the second type on $\left[v_1,\frac{v_2}{m}\right]$, we have the following inequality;;*

$$\mathbb{I}(v_1,v_2,f,w) \leq \frac{v_2-v_1}{r+1}\mathbb{J}_p\sum_{i=1}^{2}\left[|f^{(n+1)}(v_{3-i})|\mathbb{K}+m|f^{(n+1)}\left(\frac{v_i}{m}\right)|\mathbb{L}\right]^{\frac{1}{q}}, \qquad (6)$$

where $\mathbb{J}_p=\left(\int_0^1|w(t)|^p dt\right)^{\frac{1}{p}}$, $\mathbb{K}=\int_0^1 h^s\left(\frac{t}{r+1}\right)dt$ and $\mathbb{L}=\int_0^1\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt$.

Proof. As in the previous Theorem, we have

$$\begin{aligned}
&|\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)-f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt|\\
\leq &\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)|dt+\int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)|dt.
\end{aligned}$$

From Hölder's inequality, we have

$$\begin{aligned}
&\int_0^1 |w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|dt\\
\leq &\left(\int_0^1|w(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt\right)^{\frac{1}{q}}, \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v1\right)\right|dt\\
\leq &\left(\int_0^1|w(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt\right)^{\frac{1}{q}}, \qquad (8)
\end{aligned}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the (h,m)-convexity of the second kind of $\left|f^{(n+1)}\right|^q$, we obtain from (7) and (8);

$$\begin{aligned} & \int_0^1 \left| f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt \\ \leq & \left|f^{(n+1)}(v_1)\right|^q \int_0^1 h^s\left(\frac{t}{r+1}\right)dt + m\left|f^{(n+1)}\left(\frac{v_2}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt, \end{aligned} \tag{9}$$

and

$$\begin{aligned} & \int_0^1 \left| f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt \\ \leq & \left|f^{(n+1)}(v_2)\right|^q \int_0^1 h^s\left(\frac{t}{r+1}\right)dt + m\left|f^{(n+1)}\left(\frac{v_1}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \end{aligned} \tag{10}$$

Substituting (9), (10) into (7) and (8), we obtain the desired inequality. □

Theorem 2.5. *Let f be a positive real function defined on $[v_1, v_2] \subset \mathbb{R}$, such that $f^{(n+1)} \in L_1(v_1, mv_2)$. If $|f^{(n+1)}|^q$, $q > 1$, is a modified (h, m)-convex function of the second type in $\left[v_1, \frac{v_2}{m}\right]$, we have the following inequality;*

$$\mathbb{I}(v_1, v_2, f, w) \leq \frac{v_2-v_1}{r+1}\mathbb{J}_q \sum_{i=1}^{2}\left[\left(|f^{(n+1)}(v_{3-i})|^q\mathbb{G} + m|f^{(n+1)}\left(\frac{v_i}{m}\right)|^q\mathbb{H}\right)^{\frac{1}{q}}\right], \tag{11}$$

where $\mathbb{J}_q = \left(\int_0^1 |w(t)|dt\right)^{1-\frac{1}{q}}$.

Proof. We know that

$$\begin{aligned} & |\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right) - f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt\,| \\ \leq & \int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)|dt + \int_0^1 |w(t)||f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)|dt. \end{aligned}$$

Using the mean power inequality, we obtain

$$\begin{aligned} & \int_0^1 |w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|dt \\ \leq & \left(\int_0^1 |w(t)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt\right)^{\frac{1}{q}}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} & \int_0^1 |w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v1\right)\right|dt \\ \leq & \left(\int_0^1 |w(t)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Using the modified (h, m)-convexity of $|f^{(n+1)}|^q$, we have

$$\int_0^1 \left|f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt \leq |f^{(n+1)}(v_1)|^q\mathbb{G} + m||f^{(n+1)}\left(\frac{v_2}{m}\right)|^q\mathbb{H}, \tag{14}$$

and

$$\int_0^1 \left| f^{(n+1)} \left(\frac{t}{r+1} v_2 + \frac{r+1-t}{r+1} v_1 \right) \right|^q dt \leq |f^{(n+1)}(v_2)|^q \mathbb{G} + m|f^{(n+1)} \left(\frac{v_1}{m} \right) |^q \mathbb{H}. \tag{15}$$

Substituting (14) and (15) into (12) and (13) respectively, we arrive at (11). □

Remark 2.6. All previous results contain many of those reported in the literature, particularizing the function w as well as for different notions of convexity.

References

[1] P. M. Guzmán, J. E. Nápoles V., Y. Gasimov, Integral inequalities within the framework of generalized fractional integrals, Fractional Differential Calculus, Volume 11, Number 1 (2021), 69-84 doi:10.7153/fdc-2021-11-05

[2] J. E. Nápoles Valdés, F. Rabossi, A. D. Samaniego, Convex functions: Ariadne's thread or Charlotte's Spiderweb?, Advanced Mathematical Models & Applications Vol.5, No.2, 2020, pp.176-191

[3] J. E. Nápoles Valdés, J. M. Rodríguez, J. M. Sigarreta, On Hermite-Hadamard type inequalities for non-conformable integral operators, Symmetry **2019**, *11*, 1108.

[4] M. E. Özdemir, S. S. Dragomir, Ç Yıldız, The Hadamard's inequality for convex function via fractional integrals, Acta Math. Sci., 2013, 33B (5), 1293-1299.

e-mail: guzmanpaulomatias@gmail.com
e-mail: paulomatiasguzman@hotmail.com